

# Algorithms, unaffected by the Schwarz paradox, approximating tangent planes and area of smooth surfaces via inscribed triangular polyhedra

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# Chapter 1

## Introduction

### 1.1 Goals

In this work we provide algorithms<sup>1</sup> approximating the bivector  $\partial_{\ell_1}s(x) \wedge \partial_{\ell_2}s(x)$  and the integral  $\int_P |\partial_{\ell_1}s(x) \wedge \partial_{\ell_2}s(x)| dx$  of a smooth map  $s : \Omega \rightarrow \mathbb{E}_n$  (that we loosely call ‘surface’), where

- $\mathbb{E}_n$  is an  $n$ -dimensional Euclidean space;
- $\Omega$  is an open subset of the Euclidean plane  $\mathbb{E}_2$ ;
- $P \subset \Omega$  is a compact polygon;
- for every  $v \in \mathbb{E}_2$ ,  $\partial_v s(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [s(x + \epsilon v) - s(x)]$ ;
- $\{\ell_1, \ell_2\}$  is an orthonormal basis in  $\mathbb{E}_2$ ;
- $\wedge$  is the outer product in the Euclidean Clifford algebra  $\mathbb{G}_n$  associated<sup>2</sup> to  $\mathbb{E}_n$ .

In particular, if  $\partial_{\ell_1}s(x) \wedge \partial_{\ell_2}s(x) \neq 0$ , then the bivector  $\partial_{\ell_1}s(x) \wedge \partial_{\ell_2}s(x)$  can represent<sup>3</sup> the direction of the tangent plane to the surface  $s$  at point  $s(x)$  (or the normal vector, if  $s : \Omega \rightarrow \mathbb{E}_3$  and if we consider<sup>4</sup> the cross product  $\partial_{\ell_1}s(x) \times \partial_{\ell_2}s(x)$ ).

Our algorithms use informations from triangles in  $\mathbb{E}_n$  inscribed<sup>5</sup> in the surface  $s$ . Thus, Algorithm (6.4) allows to recover the tangent plane direction from every sequence of inscribed triangles converging to the point  $s(x)$ ; this result is obtained approximating<sup>6</sup> Jacobian determinants of smooth transformations  $f : \Omega \rightarrow \mathbb{E}_2$  at points  $x \in \Omega$  through nondegenerate triangles converging to point  $x$ . Algorithm (6.4) can also estimate the norm of  $\partial_{\ell_1}s(x) \wedge \partial_{\ell_2}s(x)$ , and thus, when  $s$  is globally injective, Algorithm (6.7) can

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<sup>1</sup>See (6.4) in Theorem 6.7 and (6.7) in Theorem 6.8.

<sup>2</sup>See Sections 2.3 and 2.4.

<sup>3</sup>See Section 3.1

<sup>4</sup>See Section 2.9.

<sup>5</sup>This means that the vertices of the triangles are images  $s(x)$  of vertices of some nondegenerate triangles in  $\Omega$  (see also Section 5.1).

<sup>6</sup>See (6.1) in Theorem 6.1.

approximate the area of portions of the surface  $s$  from every sequence of inscribed triangular<sup>7</sup> polyhedra uniformly convergent to that portion<sup>8</sup>.

In particular, we apply Algorithm (6.4) to the triangulation of a circular cylinder of the famous Schwarz<sup>9</sup> area paradox<sup>10</sup>, showing that the approximating inscribed balanced mean bivectors<sup>11</sup> do converge to the tangent bivectors without any restriction of the approximating triangular mesh.

As a matter of fact, by using Algorithms (6.4) and (6.7) we can restore analogies<sup>12</sup> between the limit vector  $\dot{c}_{(X)}$  of a smooth curve  $c : I \rightarrow \mathbb{E}_n$  and the limit bivector  $\partial_{\ell_1} s_{(x)} \wedge \partial_{\ell_2} s_{(x)}$  of a smooth surface  $s : \Omega \rightarrow \mathbb{E}_n$ ; such analogies are lost, according to the Schwarz paradox, if we try to approximate tangents or surface area via the usual algorithms applied to arbitrary inscribed triangular polyhedra.

## 1.2 Warnings

The aim of this work is to describe Algorithms (6.4) and (6.7) as simply as possible; thus, our intention here is not to provide the most general hypothesis under which such algorithms work; neither do we want to generalize them here to  $k$ -manifolds immersed in  $n$ -dimensional Euclidean spaces, or to Riemann manifolds, nor do we want to introduce a Stieltjes-like  $k$ -measure in  $\mathbb{E}_n$  generalizing Theorem 6.1. Such generalizations will be examined in forthcoming works.

The main theorems are stated and proved using Geometric Algebra. However, the reader will be provided formulas to translate them into the lengthy Cartesian coordinate formalism.

Finally, we apologize if some calculations may appear tedious or pedantic to readers well acquainted with Geometric Algebra, but this work is addressed to a broader audience.

## 1.3 Notations I

In this work we consider it important to distinguish the different types of mathematical objects in our formulas; therefore, we use the following conventions:

- lower-case Greek letters stand for real numbers;
- lower-case Latin letters stand for vectors in some Euclidean space  $\mathbb{E}_n$  (with the exceptions of letters  $i, j, k, m, n$ , representing integer indexes);
- capital Latin letters stand for bivectors or generic  $k$ -vectors;
- capital Greek letters stand for sets;
- capital bold Greek letters stand for  $n$ -uples or arrays of real numbers (with  $n > 1$ ).

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<sup>7</sup>This means that all faces of the polyhedron are inscribed triangles.

<sup>8</sup>See Remark 6.3.

<sup>9</sup>Hermann Amandus Schwarz (1843-1921).

<sup>10</sup>See Chapter 7.

<sup>11</sup>See Section 5.1.

<sup>12</sup>Compare, for instance, Proposition 4.1 and Theorem 6.7.

## 1.4 Historical notes

As two distinct points on a sufficiently smooth curve converge to the same point, the line passing through those two points assumes a well defined position. In particular, when such a local phenomenon is globally injective and uniform, we can approximate the length of the curve by the lengths of the line segments joining a finite number of consecutive points on the curve. The idea that a similar phenomenon may occur to triangles inscribed in a sufficiently smooth surface is probably what suggested to Serret<sup>13</sup> (see [21]) the following definition of area<sup>14</sup>:

*Soit une portion de surface courbe terminée par un contour  $C$ ; nous nommerons aire de cette surface la limite  $S$  vers laquelle tend l'aire d'une surface polyédrale inscrite formée de faces triangulaires et terminée par un contour polygonal  $F$  ayant pour limite le contour  $C$ .*

However, on 20 December 1880, Schwarz wrote to Genocchi<sup>15</sup> (see [2]) observing that the area of a curved surface cannot be defined as Serret did. In subsequent letters to Genocchi, Schwarz showed that even the area of a surface as simple as a bounded part of a right circular cylinder cannot be recovered using Serret's definition. Schwarz even provided examples of sequences of inscribed triangular polyhedra whose areas converge to any given number not less than the area of the cylinder (and even to infinity) as the polyhedra approach uniformly the cylinder<sup>16</sup>. Such phenomenon, that may occur to every curved surface (and even to polyhedra<sup>17</sup>) is given the name of **Schwarz paradox** (or **Schwarz phenomenon**).

That famous paradox apparently destroyed the possibility of defining the area of a smooth surface by analogy with the length of a smooth curve. Besides, the local interpretation of the Schwarz phenomenon implies that as three noncollinear points on a smooth surface converge to the same point of the surface, the limit position of the plane passing through those three points is not well determined, and can differ from the tangent plane to the surface at the limiting point. Also, Schwarz's counterexample shows that the limiting position of the secant plane can even be orthogonal to the actual tangent plane.

Two questions naturally arise:

- what sequences of inscribed triangular polyhedra approaching a surface have areas converging to the area of that surface ?
- are there algorithms able to recover the area of a surface from every sequence of inscribed triangular polyhedra approaching that surface ?

Schwarz showed that those questions are not trivial even for a cylinder.

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<sup>13</sup>Joseph Alfred Serret (1819-1885).

<sup>14</sup>Our translation: "Let a portion of a curved surface be bounded by a contour  $C$ ; we will call area of that surface the limit  $S$  to which converges the area of an inscribed polyhedral surface whose faces are triangles and which is bounded by a polygonal contour  $F$  having  $C$  as limit."

<sup>15</sup>Angelo Genocchi (1817-1889).

<sup>16</sup>As a consequence, there also exist sequences of inscribed triangular polyhedra approaching the cylinder whose areas have no limit.

<sup>17</sup>See [7].

Many different approaches were used to answer those questions. We cannot summarize such a long and prolific history here<sup>18</sup>; we will just focus on some particular issues which only concern smooth curved surfaces.

- Apart from Peano<sup>19</sup>, all authors<sup>20</sup> approximated the area of a curved surface using the areas of triangular polyhedra uniformly approaching the surface.
  - Most of those authors selected particular inscribed triangular polyhedra constraining the form or the position of the triangular faces with ‘ad hoc’ conditions.
  - Lebesgue<sup>21</sup>, instead, freed himself from inscribed polyhedra and artificial geometric conditions; however, his definition of area<sup>22</sup> is of no help in selecting a sequence of polyhedra whose areas converge to the area of the surface<sup>23</sup>, nor does his definition of area correspond locally to a definition of tangent.
  - Geöcze<sup>24</sup> conjectured<sup>25</sup>, and Mulholland<sup>26</sup> proved in [15], that Lebesgue’s area can also be obtained restricting Lebesgue’s approach to inscribed polyhedra.
- Peano freed himself from polyhedra<sup>27</sup>, and used his *Calcolo Geometrico* (based on Grassmann exterior algebra<sup>28</sup>) to define the area through integrals taken on the boundaries of portions of a surface. However, his definition<sup>29</sup> was vague about what portions of a surface may cut in order to approximate the area of the whole surface.

Our Algorithm (6.7) allows to consider inscribed triangular polyhedra without any kind of constraint, and uses a slightly modified notion of area. Besides, Algorithm (6.7) is just a global adaptation of the local Algorithm (6.4) that approximates tangent planes from every inscribed triangle approaching a point on the surface. Thus, Algorithms (6.4) and (6.7) restore many of the analogies between curves and surfaces.

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<sup>18</sup>Suggested readings are [8], [3], [19] and [20].

<sup>19</sup>Giuseppe Peano (1858-1932).

<sup>20</sup>Another exception is William Henry Young (1863-1942), who used an approach similar to Peano’s in [23].

<sup>21</sup>Henri Lebesgue (1875-1941).

<sup>22</sup>See [11]

<sup>23</sup>We suggest to read the Jordan’s criticism to Lebesgue’s definition of area in [12] at pages 163–164.

<sup>24</sup>Zoárd Geöcze (1873-1916).

<sup>25</sup>See [19].

<sup>26</sup>H. P. Mulholland (we did not find any biographical data about him).

<sup>27</sup>Nevertheless, his work has been a key inspiration to us.

<sup>28</sup>See [16] and [17].

<sup>29</sup>See [16] on page 164, or [18] on page 55.



## Chapter 2

# Basic notions of Euclidean Clifford algebras

### 2.1 Motivations

Theorem 6.1, Theorem 6.7 and Theorem 6.8 are stated and proved using Euclidean Clifford algebra (i.e. **Geometric Algebra**). Of course, they can be translated into the Cartesian coordinatewise language as well<sup>1</sup>; however, we consider the coordinate-free language of Geometric Algebra to be richer and more suitable in order to algebraically represent geometric properties. Moreover, we discovered Algorithm (6.4) and (6.7) while exploring the Schwarz paradox via Geometric Algebra and not via Cartesian language.

### 2.2 Formal Geometric Algebra

Following is a brief formal description of Geometric Algebra  $\mathbb{G}_n$ . For more details and other approaches, see also [1], [4], [5], [6], [9], [10], [13], [14] or [22].

Suppose we have an ordered alphabet of  $n$  (distinct) letters  $\mathcal{A}_n = \{\ell_1, \dots, \ell_n\}$ . A **word** from this alphabet is a juxtaposition of letters taken from  $\mathcal{A}_n$ . A word with no letters is considered a word as well, it is named **empty word** (or **unit**), and it is given the reserved<sup>2</sup> symbol  $\mathbb{1}$ .

The set of formal finite real linear combinations of words<sup>3</sup> from  $\mathcal{A}_n$  forms a real algebra  $\mathbb{G}_n$  if we consider juxtaposition of words as an associative and distributive product among words<sup>4</sup>. Thus,  $\mathbb{1}$  is the unit for that product.

Also the empty real linear combination of words is considered an element of such algebra, and it is given the symbol  $\mathbb{0}$ . The following axioms hold in  $\mathbb{G}_n$ :

$$\ell_j \neq \mathbb{1}, \quad \ell_j \neq \mathbb{0}, \quad \mathbb{0} \neq \mathbb{1}, \quad 0W = \mathbb{0}, \quad 1W = W,$$

where  $j = 1, \dots, n$  and  $W$  is a word from the alphabet  $\mathcal{A}_n$ ; moreover,

$$\ell_i \ell_j = \begin{cases} -\ell_j \ell_i & \text{if } i \neq j, \\ \mathbb{1} & \text{if } i = j, \end{cases} \quad (2.1)$$

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<sup>1</sup>We provide formulas to do it.

<sup>2</sup>A symbol is called ‘reserved’ if it can never be a letter of any alphabet.

<sup>3</sup>We will write real coefficients on left of words.

<sup>4</sup>The real coefficients are multiplied among themselves in  $\mathbb{R}$ .

where  $-\ell_j \ell_i$  abbreviates  $(-1)\ell_j \ell_i$ . The complete ordered word  $\ell_1 \ell_2 \cdots \ell_n$  is called **pseudo-unit** and is given the reserved symbol  $\mathbb{I}_n$ .  $\mathbb{G}_n$  is then uniquely determined<sup>5</sup> if we add the final axioms

$$\mathbb{I}_n \neq \mathbb{O} , \quad \mathbb{I}_n \neq \mathbb{1} , \quad \mathbb{I}_n \neq -\mathbb{1} .$$

Axiom (2.1) allows to reduce every nonempty word from the alphabet  $\mathcal{A}_n$  to a unique minimal<sup>6</sup> ordered word (with sign)

$$\pm \ell_{i_1} \ell_{i_2} \cdots \ell_{i_k} ,$$

where  $i_1 < i_2 < \cdots < i_k$ . The number  $k$  is called **grade** of the word, and the sign is called **orientation** of the word (with respect to the ordered alphabet  $\mathcal{A}_n$ ). Such reductions make  $\mathbb{G}_n$  a graded algebra

$$\mathbb{G}_n = \bigoplus_{k=0}^n \mathbb{G}_{(n)}^{(k)} ,$$

where  $\mathbb{G}_{(n)}^{(k)}$  is the linear subspace of (finite) real combinations of words of grade  $k$  (notice that  $\mathbb{G}_{(n)}^{(k)}$  is not a subalgebra). Each  $\mathbb{G}_{(n)}^{(k)}$  has (real) dimension  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ , and  $\mathbb{G}_n$  has dimension  $2^n$ .

We can unambiguously identify the algebra of real numbers  $\mathbb{R}$  with  $\mathbb{G}_{(n)}^{(0)}$ , the real number 1 with unit  $\mathbb{1}$ , and  $0 \in \mathbb{R}$  with the empty linear combination  $\mathbb{O}$ .

### 2.3 Euclidean structure of $\mathbb{G}_n$

Geometric Algebra  $\mathbb{G}_n$  is also called **Euclidean Clifford algebra**, because it possesses a Euclidean structure strictly tied with its algebraic product<sup>7</sup>. As a matter of fact, the symmetric part of the product among elements  $x, y \in \mathbb{G}_{(n)}^{(1)}$

$$\frac{1}{2}(xy + yx)$$

is always a real number and, as a function of  $x$  and  $y$ , it is a symmetric, positive definite bilinear form in  $\mathbb{G}_{(n)}^{(1)}$  (that we denote with the symbol  $\mathbf{x} \cdot \mathbf{y}$ ).

The  $n$  letters of the ordered alphabet  $\mathcal{A}_n$  (generating  $\mathbb{G}_n$ ) form an ordered orthonormal basis in  $\mathbb{G}_{(n)}^{(1)}$  with respect to the scalar product  $x \cdot y$ , indeed

$$\ell_i \cdot \ell_j = \frac{1}{2}(\ell_i \ell_j + \ell_j \ell_i) = \begin{cases} 0 & \text{if } i \neq j , \\ 1 & \text{if } i = j . \end{cases}$$

It is also important to note that the antisymmetric part of the product between  $x, y \in \mathbb{G}_{(n)}^{(1)}$

$$\frac{1}{2}(xy - yx)$$

<sup>5</sup>Unique up to algebra isomorphisms between real associative algebras with unit.

<sup>6</sup>That is, without repeated letters.

<sup>7</sup>And because it was introduced by William Kingdon Clifford (1845-1879); see [4].

is always an element of  $\mathbb{G}_{\binom{n}{2}}$ , it is given the symbol  $\mathbf{x} \wedge \mathbf{y}$ , and is called **outer product**<sup>8</sup> because it acts in the graded algebra  $\mathbb{G}_n$  as the associative antisymmetric Grassmann exterior product acts on the graded Grassmann algebra  $\bigoplus_{k=0}^n \left[ \bigwedge^k \mathbb{G}_{\binom{n}{1}} \right]$ . So, the product of  $x, y \in \mathbb{G}_{\binom{n}{1}}$  can be decomposed as

$$xy = (x \cdot y) + (x \wedge y) ,$$

and

$$\ell_i \ell_j = (\ell_i \cdot \ell_j) + (\ell_i \wedge \ell_j) = \begin{cases} \ell_i \wedge \ell_j = -\ell_j \wedge \ell_i & \text{if } i \neq j , \\ 1 & \text{if } i = j . \end{cases}$$

## 2.4 The Geometric Algebra associated to an oriented Euclidean space

Here we notice that the construction of  $\mathbb{G}_n$  can proceed the other way around as well: given a  $n$ -dimensional Euclidean space  $\mathbb{E}_n$ , there exists a unique<sup>9</sup> Geometric Algebra  $\mathbb{G}_n$  such that its Euclidean subspace  $\mathbb{G}_{\binom{n}{1}}$  is isometric to  $\mathbb{E}_n$ . Indeed, it suffices to choose an ordered orthonormal basis  $\{e_1, e_2, \dots, e_n\} \subset \mathbb{E}_n$  as the ordered alphabet generating  $\mathbb{G}_n$ . In this sense we speak of Geometric Algebra associated to the oriented<sup>10</sup> Euclidean space  $\mathbb{E}_n$ .

A fundamental improvement of Geometric Algebra  $\mathbb{G}_n = \bigoplus_{k=0}^n \mathbb{G}_{\binom{n}{k}}$  over Grassmann algebra  $\bigoplus_{k=0}^n \left[ \bigwedge^k \mathbb{E}_n \right]$  is that  $\mathbb{G}_n$  has a well defined Euclidean structure such that

- each subspace  $\mathbb{G}_{\binom{n}{k}}$  is orthogonal in  $\mathbb{G}_n$  to every other  $\mathbb{G}_{\binom{n}{j}}$ , with  $k \neq j$ ;
- each subspace  $\mathbb{G}_{\binom{n}{k}}$  has a Euclidean structure, i.e. a symmetric, positive-definite bilinear form (that we will continue to indicate with the dot  $\cdot$ ) uniquely determined by the scalar product in  $\mathbb{G}_{\binom{n}{1}}$  (usually identified with  $\mathbb{E}_n$ ).

For instance, for each  $a, b, c, d \in \mathbb{G}_{\binom{3}{1}} \equiv \mathbb{E}_3$

$$(a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

is the scalar product  $(a \wedge b) \cdot (c \wedge d)$  in  $\mathbb{G}_3$  restricted to the subspace  $\mathbb{G}_{\binom{3}{2}}$ .

Notice that the one-dimensional subspace  $\mathbb{G}_{\binom{3}{0}}$  (that we identified with  $\mathbb{R}$ ) has the usual product between real numbers as the restriction of the scalar product in  $\mathbb{G}_3$

$$(\alpha \mathbf{1}) \cdot (\beta \mathbf{1}) = \alpha\beta = (\alpha \mathbf{1})(\beta \mathbf{1}),$$

while

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<sup>8</sup>Or, simply, exterior product.

<sup>9</sup>Up to isometries and orientation.

<sup>10</sup>The orientation being determined by the order of the orthonormal basis.

$$(\alpha \mathbb{I}_n) \cdot (\beta \mathbb{I}_n) = \alpha\beta, \text{ and } (\alpha \mathbb{I}_n)(\beta \mathbb{I}_n) = (-1)^{\frac{n(n-1)}{2}} \alpha\beta.$$

Roughly speaking,  $\mathbb{G}_n$  encodes the scalar product of  $\mathbb{E}_n$ , its orientation<sup>11</sup>, and the Grassmann exterior product on  $\bigoplus_{k=0}^n \left[ \bigwedge^k \mathbb{E}_n \right]$ , within its associative and distributive algebraic product; moreover, such encoding reveals many connections between algebra and geometry, making new insights possible.

## 2.5 Notations II

As we have already said, the Geometric Algebra  $\mathbb{G}_n$  is a Euclidean space; we indicate the norm of  $X \in \mathbb{G}_n$  with symbol  $|\mathbf{X}| = \sqrt{X \cdot X}$ .

In order to emphasize the geometric interpretation of elements in a Geometric algebra, we will use the following nomenclature.

Given an orthonormal ordered basis in the  $n$ -dimensional Euclidean space  $\mathbb{E}_n$  (or an ordered alphabet)  $\{\ell_1, \dots, \ell_n\}$ , then

- elements of  $\mathbb{G}_{\binom{n}{0}} \equiv \mathbb{R}$  are called **scalars**,
- elements of  $\mathbb{G}_{\binom{n}{1}} \equiv \mathbb{E}_n$  are called **vectors**,
- elements of  $\mathbb{G}_{\binom{n}{2}}$  are called **bivectors**,
- elements of  $\mathbb{G}_{\binom{n}{k}}$  are called  **$k$ -vectors**,
- elements of  $\mathbb{G}_{\binom{n}{n}} \equiv \mathbb{R}$  are called **pseudo-scalars**, and are real multiples of the pseudo-unit  $\mathbb{I}_n = \ell_1 \cdots \ell_n$ .

A  $k$ -vector of the form  $\alpha(v_1 \wedge \cdots \wedge v_k)$ , where  $\alpha \in \mathbb{R}$  and each  $v_i \in \mathbb{G}_{\binom{n}{1}}$ , is called  **$k$ -blade**. Note that in  $\mathbb{G}_3$  every bivector is a 2-blade, while in  $\mathbb{G}_4$  the bivector  $(\ell_1 \ell_2) + (\ell_3 \ell_4)$  is not a 2-blade.

In order to limit the use of parentheses, we establish the following precedence rules for operations in  $\mathbb{G}_n$ , listed below with decreasing rank of precedence<sup>12</sup>:

1. outer product  $\wedge$ ,
2. product in  $\mathbb{G}_n$  among elements of the same grade,
3. scalar product,
4. product between a scalar and a  $k$ -vector (with  $k > 0$ ),
5. sum in  $\mathbb{G}_n$ .

Thus, for example,  $\alpha a + \beta b$  means  $(\alpha a) + (\beta b)$ ;  $\alpha\beta x \wedge y$  means  $(\alpha\beta)(x \wedge y)$ . However, note also that  $\alpha\beta x \wedge y = (\alpha x) \wedge (\beta y) = (\beta x) \wedge (\alpha y) = x \wedge (\alpha\beta y) = \dots$

<sup>11</sup>Encoded by its pseudo-unit  $\mathbb{I}_n$ .

<sup>12</sup>Sometimes we will also use spacing to stress precedence.

## 2.6 Inverse of vectors and pseudo-unit in $\mathbb{G}_n$

In  $\mathbb{G}_n$  a vector  $v$  is invertible if and only if  $v \neq 0$ ; in this case we have

$$v^{-1} = \frac{1}{|v|^2} v ,$$

where  $|v| = \sqrt{v \cdot v} = \sqrt{v^2}$ . In fact  $vv^{-1} = \frac{1}{|v|^2} vv = \frac{1}{|v|^2} (v \cdot v + v \wedge v) = 1$ .

In  $\mathbb{G}_n$  the pseudo-unit is always invertible, and

$$(\mathbb{I}_n)^{-1} = (\ell_1 \cdots \ell_n)^{-1} = \ell_n \cdots \ell_1 = (-1)^{\frac{n(n-1)}{2}} \mathbb{I}_n .$$

## 2.7 Pseudo-scalars in $\mathbb{G}_2$ and determinants of $2 \times 2$ real matrices

In  $\mathbb{G}_2$  the notions of bivector, 2-blade and pseudo-scalar coincide.

Let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in  $\mathbb{E}_2 \equiv \mathbb{G}_{(2)}^{(1)}$ .

For each  $x, y \in \mathbb{G}_{(2)}^{(1)}$ , we can write  $x = \chi_1 \ell_1 + \chi_2 \ell_2$  and  $y = \zeta_1 \ell_1 + \zeta_2 \ell_2$  (where  $\chi_i = x \cdot \ell_i$  and  $\zeta_i = y \cdot \ell_i$ ), then

$$x \wedge y = (\chi_1 \zeta_2 - \chi_2 \zeta_1) (\ell_1 \wedge \ell_2) = \det \begin{pmatrix} \chi_1 & \chi_2 \\ \zeta_1 & \zeta_2 \end{pmatrix} \ell_1 \ell_2 = \det \begin{pmatrix} \chi_1 & \chi_2 \\ \zeta_1 & \zeta_2 \end{pmatrix} \mathbb{I}_2 ,$$

and then

$$(x \wedge y)(\mathbb{I}_2)^{-1} = \det \begin{pmatrix} \chi_1 & \chi_2 \\ \zeta_1 & \zeta_2 \end{pmatrix} = (x \wedge y) \cdot \mathbb{I}_2 .$$

Notice that the one-dimensional space  $\mathbb{G}_{(2)}^{(1)}$  has the following positive definite symmetric bilinear form:

$$(x \wedge y) \cdot (w \wedge z) = (x \cdot w)(y \cdot z) - (x \cdot z)(y \cdot w) ,$$

where  $x, y, w, z \in \mathbb{G}_{(2)}^{(1)}$ .

It will be useful to note the following property, too.

**Proposition 2.1.** *If  $x, y \in \mathbb{E}_2$ , then  $|x \wedge y| \leq |x| |y|$ .*

**Proof:** let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in  $\mathbb{E}_2$ ; let us write  $x = \chi_1 \ell_1 + \chi_2 \ell_2$  and  $y = \zeta_1 \ell_1 + \zeta_2 \ell_2$ , so that

$$|x \wedge y|^2 = (\chi_1 \zeta_2 - \chi_2 \zeta_1)^2 .$$

The thesis is then achieved verifying the following equivalence,

$$(\chi_1 \zeta_2 - \chi_2 \zeta_1)^2 \leq (\chi_1^2 + \chi_2^2)(\zeta_1^2 + \zeta_2^2) \iff 0 \leq (\chi_1 \zeta_1 + \chi_2 \zeta_2)^2 . \square$$

In this work the symbol  $\square$  indicates the end of a proof.

## 2.8 Exterior product of orthogonal vectors in $\mathbb{E}_2$

**Proposition 2.2.** *If  $x, y \in \mathbb{E}_2$  are orthogonal, then*

$$x \wedge y = \begin{cases} |x| |y| \mathbb{I}_2 & \text{if } (x \wedge y) \cdot \mathbb{I}_2 > 0 \\ -|x| |y| \mathbb{I}_2 & \text{if } (x \wedge y) \cdot \mathbb{I}_2 < 0 \end{cases}$$

**Proof:** let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in  $\mathbb{E}_2$ ;  $x \wedge y = [(x \wedge y) \cdot \mathbb{I}_2] \mathbb{I}_2$ ; let us write  $x = \chi_1 \ell_1 + \chi_2 \ell_2$  and  $y = \zeta_1 \ell_1 + \zeta_2 \ell_2$ , so that  $x \wedge y = (\chi_1 \zeta_2 - \chi_2 \zeta_1) \mathbb{I}_2$ , then

$$|\chi_1 \zeta_2 - \chi_2 \zeta_1|^2 = (\chi_1)^2 (\zeta_2)^2 + (\chi_2)^2 (\zeta_1)^2 - \chi_1 \zeta_1 \chi_2 \zeta_2 - \chi_1 \zeta_1 \chi_2 \zeta_2 ,$$

and orthogonality corresponds to the relation  $\chi_1 \zeta_1 = -\chi_2 \zeta_2$ , so

$$|\chi_1 \zeta_2 - \chi_2 \zeta_1|^2 = (\chi_1)^2 (\zeta_2)^2 + (\chi_2)^2 (\zeta_1)^2 + (\chi_1)^2 (\zeta_1)^2 + (\chi_2)^2 (\zeta_2)^2 = |x|^2 |y|^2 \quad \square$$

Two ordered bases<sup>13</sup>  $\{b_1, b_2\}$  and  $\{c_1, c_2\}$  in  $\mathbb{E}_2$  are said to be **equi-oriented** if

$$(b_1 \wedge b_2) \cdot (c_1 \wedge c_2) > 0.$$

## 2.9 Isometric duality between $\mathbb{G}_{\binom{3}{1}}$ and $\mathbb{G}_{\binom{3}{2}}$ : the cross product

In  $\mathbb{G}_3$  the subspaces  $\mathbb{G}_{\binom{3}{1}}$  and  $\mathbb{G}_{\binom{3}{2}}$  have the same dimension, and are both Euclidean spaces. Moreover, the correspondence

$$\mathbb{G}_{\binom{3}{1}} \ni x \longmapsto \mathbf{x}^* = x \mathbb{I}_3 \in \mathbb{G}_{\binom{3}{2}} ,$$

is an isometry in  $\mathbb{G}_3$ , whose inverse is the correspondence

$$\mathbb{G}_{\binom{3}{2}} \ni X \longmapsto \mathbf{X}^\# = -X \mathbb{I}_3 \in \mathbb{G}_{\binom{3}{1}} .$$

This allows to establish the classical 1 : 1 correspondence between a two-dimensional vector subspace of  $\mathbb{E}_3$  generated by the two independent vectors  $a, b \in \mathbb{G}_{\binom{3}{1}}$  (i.e. the bivector  $a \wedge b \in \mathbb{G}_{\binom{3}{2}}$ ) and the vector  $(a \wedge b)^\# \in \mathbb{G}_{\binom{3}{1}}$  that is orthogonal to that subspace; as a matter of fact

$$[(a \wedge b)^\#] \cdot a = -[(a \wedge b) \mathbb{I}_3] \cdot a = -\frac{1}{4}[(ab - ba) \mathbb{I}_3 a + a(ab - ba) \mathbb{I}_3] = 0 ,$$

as  $w \mathbb{I}_3 = \mathbb{I}_3 w$  for all  $w \in \mathbb{G}_{\binom{3}{1}}$  ( $[(a \wedge b)^\#] \cdot b = 0$ , analogously). In this sense, the application

$$\begin{aligned} \mathbb{G}_{\binom{3}{1}} \times \mathbb{G}_{\binom{3}{1}} &\longrightarrow \mathbb{G}_{\binom{3}{1}} \\ (a, b) &\longmapsto (a \wedge b)^\# = -(x \wedge y) \mathbb{I}_3 , \end{aligned}$$

corresponds to the classical cross product  $a \times b$ .

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<sup>13</sup>Or, what is the same think, two 2-blades  $b_1 \wedge b_2, c_1 \wedge c_2$  in  $\mathbb{E}_2$ .

## Chapter 3

# Geometric Algebra and Geometry

### 3.1 Point, lines and planes in $\mathbb{E}_n$

In an  $n$ -dimensional real affine Euclidean space  $\mathbb{A}_n$ , if one fixes a point as the origin, the points in  $\mathbb{A}_n$  can be identified with vectors in a Euclidean space  $\mathbb{E}_n$ . With respect to the foregoing identification, we will talk about points, lines and planes in a Euclidean space; that is, vectors in  $\mathbb{E}_n$  are considered as points of an affine Euclidean space where some point (the corresponding origin  $\mathbb{O}$ ) is fixed somewhere<sup>1</sup>.

An **oriented (linear) direction** in  $\mathbb{E}_n$  is a nonzero vector.

An **oriented (planar) direction** in  $\mathbb{E}_n$  is a nonzero 2-blade in  $\mathbb{G}_n$  (the Geometric Algebra associated to  $\mathbb{E}_n$ ).

A **line** in  $\mathbb{E}_n$  (passing through point  $x_0 \in \mathbb{E}_n$  and parallel to the oriented direction  $v \in \mathbb{E}_n$ ) is the set

$$\{x \in \mathbb{E}_n : \exists \tau \in \mathbb{R} \quad x = x_0 + \tau v\} .$$

Considering  $\mathbb{E}_n \equiv \mathbb{G}_{\binom{n}{1}}$ , the same set can be described through vectors and 2-blades as follows:

$$\{x \in \mathbb{G}_{\binom{n}{1}} : (x - x_0) \wedge v = 0\} .$$

Similarly, a **plane** in  $\mathbb{E}_n$  (passing through point  $x_0 \in \mathbb{E}_n$  and parallel to the vector space generated by the two independent vectors  $u, v \in \mathbb{E}_n$ ) is the set

$$\{x \in \mathbb{E}_n : \exists \mu, \nu \in \mathbb{R} \quad x = x_0 + \mu u + \nu v\} ,$$

which can also be described through vectors and 3-vectors in  $\mathbb{G}_n$  as follows:

$$\{x \in \mathbb{G}_{\binom{n}{1}} : (x - x_0) \wedge u \wedge v = 0\} ,$$

characterized by point  $x_0 \in \mathbb{E}_n \equiv \mathbb{G}_{\binom{n}{1}}$  and the oriented (plane) direction  $u \wedge v \in \mathbb{G}_{\binom{n}{2}}$ .

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<sup>1</sup>See, for instance, ...

### 3.2 Oriented intervals and triangles in $\mathbb{E}_n$

Geometric Algebra is particularly suited to deal with triangles in Euclidean spaces. In particular, the coordinate-free formalism of Geometric Algebra provides analogies between intervals in  $\mathbb{R}$  and triangles in  $\mathbb{E}_n$ .

Let  $\{h_1, \dots, h_n\}$  be an ordered orthonormal basis in the  $n$ -dimensional Euclidean space  $\mathbb{E}_n$ .

An **interval** in  $\mathbb{E}_n$  with **extremities**  $a, b \in \mathbb{E}_n$  is the set

$$\{x \in \mathbb{E}_n : x = \alpha a + \beta b, \alpha + \beta = 1, \alpha, \beta \geq 0\}.$$

When we want to attribute an orientation to this set, depending on the order of its extremities, we indicate it with the ordered brackets  $[a, b]$  and call it **oriented interval** in  $\mathbb{E}_n$ . The length of an oriented interval  $[a, b]$  is, of course,  $|a - b|$ .

Analogously, a **triangle** in  $\mathbb{E}_n$  with **vertices**  $a, b, c \in \mathbb{E}_n$  is the set

$$\{x \in \mathbb{E}_n : x = \alpha a + \beta b + \gamma c, \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0\}.$$

When we want to attribute an orientation to this set, depending on the order of its vertices, we indicate it within the brackets  $[a, b, c]$  and call it **oriented triangle** in  $\mathbb{E}_n$ .

The following oriented triangles

$$[b, c, a] \qquad [c, a, b]$$

correspond to the same triangle and have the same orientation with  $[a, b, c]$ , while the oriented triangles

$$[a, c, b] \qquad [c, b, a] \qquad [b, a, c]$$

correspond to the same triangle of  $[a, b, c]$ , but have opposite orientation with respect the orientation of  $[a, b, c]$ .

The oriented intervals  $[a, b]$ ,  $[b, c]$  and  $[c, a]$  are called the **sides** of the oriented triangle  $[a, b, c]$ . A **diameter** of an oriented triangle is a side of maximal length<sup>2</sup>.

Given an oriented triangle  $[a, b, c]$ , we consider the bivector

$$\langle a; b; c \rangle = a \wedge b + b \wedge c + c \wedge a \in \mathbb{G}_{\binom{n}{2}}.$$

Such a bivector is a 2-blade, indeed. For, given the oriented triangle  $[a, b, c]$ , if we define

$$\ell_a = c - b \qquad \ell_b = a - c \qquad \ell_c = b - a,$$

then

$$\langle a; b; c \rangle = \ell_a \wedge \ell_b = \ell_b \wedge \ell_c = \ell_c \wedge \ell_a.$$

Notice that

$$\langle a; b; c \rangle = \langle b; c; a \rangle = \langle c; a; b \rangle = -\langle a; c; b \rangle = -\langle c; b; a \rangle = -\langle b; a; c \rangle,$$

---

<sup>2</sup>An oriented triangle can have more than one diameter, of course.



that is,  $\langle a; b; c \rangle$  change sign if we change the orientation of  $[a, b, c]$ .

The **area** of a triangle whose vertices are  $a, b, c$  is  $\frac{1}{2} |\langle a; b; c \rangle|$ . Moreover, such area can also be expressed using only the scalar product

$$\frac{1}{2} |\langle a; b; c \rangle| = \frac{1}{2} \sqrt{(\ell_a \wedge \ell_b) \cdot (\ell_a \wedge \ell_b)} = \frac{1}{2} \sqrt{|\ell_a|^2 |\ell_b|^2 - (\ell_a \cdot \ell_b)^2}.$$

A triangle, whose area is zero, is called **degenerate**.

Let us now express the bivector  $\langle a; b; c \rangle \in \mathbb{G}_{\binom{n}{2}}$  coordinatewise.

If  $\{h_1, \dots, h_n\}$  is an ordered orthonormal basis in the  $n$ -dimensional Euclidean space  $\mathbb{E}_n$ , and

$$a = \sum_{j=1}^n \alpha_j h_j \quad b = \sum_{j=1}^n \beta_j h_j \quad c = \sum_{j=1}^n \gamma_j h_j,$$

then

$$\begin{aligned} \langle a; b; c \rangle &= a \wedge b + b \wedge c + c \wedge a = \\ &= \sum_{1 \leq j < k \leq n} \left[ \det \begin{pmatrix} \alpha_j & \alpha_k \\ \beta_j & \beta_k \end{pmatrix} + \det \begin{pmatrix} \beta_j & \beta_k \\ \gamma_j & \gamma_k \end{pmatrix} + \det \begin{pmatrix} \gamma_j & \gamma_k \\ \alpha_j & \alpha_k \end{pmatrix} \right] h_j \wedge h_k = \\ &= (b - a) \wedge (c - a) = \sum_{1 \leq j < k \leq n} \det \begin{pmatrix} \beta_j - \alpha_j & \beta_k - \alpha_k \\ \gamma_j - \beta_j & \gamma_k - \beta_k \end{pmatrix} h_j \wedge h_k. \end{aligned}$$

In particular,

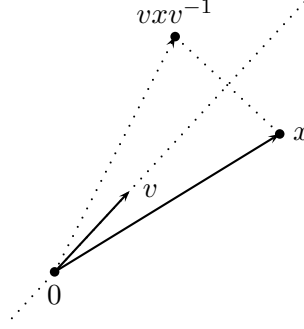
$$\frac{1}{2} |\langle a; b; c \rangle| = \frac{1}{2} \sqrt{\sum_{1 \leq j < k \leq n} \left[ \det \begin{pmatrix} \beta_j - \alpha_j & \beta_k - \alpha_k \\ \gamma_j - \beta_j & \gamma_k - \beta_k \end{pmatrix} \right]^2},$$

since  $\{h_j \wedge h_k\}_{1 \leq j < k \leq n}$  is an orthonormal basis in  $\mathbb{G}_{\binom{n}{2}}$ .

### 3.3 Reflections in $\mathbb{E}_n$ and mirror vertices in plane triangles

Invertible vectors (that is, linear directions) are useful to represent mirror points with respect to those directions.

Let us consider a point  $x \in \mathbb{E}_n$  and a direction  $v \in \mathbb{E}_n$ ; if  $x$  and  $v$  are linearly independent ( $x \wedge v \neq 0$ ), then the mirror image of  $x$  with respect to the line passing through 0 and  $v$  is the point



$$\begin{aligned}
 vxv^{-1} &= (v \cdot x + v \wedge x)v^{-1} = (v \cdot x - x \wedge v)v^{-1} = \\
 &= [v \cdot x - (xv - x \cdot v)]v^{-1} = (2v \cdot x - xv)v^{-1} = 2\frac{x \cdot v}{|v|^2}v - x
 \end{aligned}$$

(see [13] at pag.13 for further details). Note that the foregoing formula works even when  $x \wedge v = 0$  (that is,  $x = \chi v$  for some  $\chi \in \mathbb{R}$ ).

Let us consider a nondegenerate oriented triangle  $[a, b, c]$  in  $\mathbb{E}_2$  (that is, a plane triangle), then each of its (oriented) sides  $[a, b]$ ,  $[b, c]$  and  $[c, a]$  determines a direction ( $\ell_c$ ,  $\ell_a$  and  $\ell_b$  respectively). So we can consider the mirror image  $\mathbf{x}'$  of each vertex  $x$  of  $[a, b, c]$  with respect to the line passing through its two adjacent vertices. We have that

$$\begin{aligned}
 a' &= c + \ell_a \ell_b \ell_a^{-1} = c + 2\frac{\ell_a \cdot \ell_b}{|\ell_a|^2} \ell_a - \ell_b = - \left[ a + 2b\frac{\ell_b \cdot \ell_a}{|\ell_a|^2} + 2c\frac{\ell_c \cdot \ell_a}{|\ell_a|^2} \right], \\
 b' &= a + \ell_b \ell_c \ell_b^{-1} = a + 2\frac{\ell_b \cdot \ell_c}{|\ell_b|^2} \ell_b - \ell_c = - \left[ b + 2c\frac{\ell_c \cdot \ell_b}{|\ell_b|^2} + 2a\frac{\ell_a \cdot \ell_b}{|\ell_b|^2} \right], \\
 c' &= b + \ell_c \ell_a \ell_c^{-1} = b + 2\frac{\ell_c \cdot \ell_a}{|\ell_c|^2} \ell_c - \ell_a = - \left[ c + 2a\frac{\ell_a \cdot \ell_c}{|\ell_c|^2} + 2b\frac{\ell_b \cdot \ell_c}{|\ell_c|^2} \right],
 \end{aligned}$$

and we call them **mirror vertices** of the oriented<sup>3</sup> triangle  $[a, b, c]$ .

Given a nondegenerate oriented plane triangle  $[a, b, c]$ , it will be useful to define the following point and two directions

$$\bar{\mathbf{a}} = \frac{1}{2}(a' + a), \quad \mathbf{u}_a = \frac{1}{2}(a' - a), \quad \mathbf{v}_a = c - \bar{a}.$$

The above definitions can be generalized to any vertex. Indeed, if  $x$  is a vertex of a nondegenerate oriented plane triangle  $[x, x_+, x_-]$  one can define

$$\begin{aligned}
 x' &= x_- + \ell_x \ell_{x_+} \ell_x^{-1} = x_- + 2\frac{\ell_x \cdot \ell_{x_+}}{|\ell_x|^2} \ell_x - \ell_{x_+} = \\
 &= - \left[ x + 2(x_+)\frac{\ell_{x_+} \cdot \ell_x}{|\ell_x|^2} + 2(x_-)\frac{\ell_{x_-} \cdot \ell_x}{|\ell_x|^2} \right], \\
 \bar{x} &= \frac{1}{2}(x' + x), \quad u_x = \frac{1}{2}(x' - x), \quad v_x = x_- - \bar{x},
 \end{aligned}$$

where  $\ell_x = (x_- - x_+)$ ,  $\ell_{x_+} = (x - x_-)$  and  $\ell_{x_-} = (x_+ - x)$ . So we have that

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<sup>3</sup>However, they do not depend on the triangle's orientation.

$$x = \bar{x} - u_x \quad x_+ = \bar{x} + (v_x - \ell_x) \quad x_- = \bar{x} + v_x \quad x' = \bar{x} + u_x ,$$

and we can state the following proposition.

**Proposition 3.1.** *Let  $x$  be a vertex of the nondegenerate oriented plane triangle  $[x, x_+, x_-]$ , then*

1.  $2u_x \wedge \ell_x = 2\langle x; x_+; x_- \rangle = \langle x; x_+; x_- \rangle - \langle x'; x_+; x_- \rangle$  ;
2.  $u_x \cdot \ell_x = 0$  (so that  $u_x \wedge \ell_x = \pm |u_x| |\ell_x| \mathbb{I}_2$ ) ;
3.  $v_x \wedge \ell_x = 0$  (so there exists  $\tau \in \mathbb{R}$  such that  $v_x = \tau \ell_x$ ) .

**Proof of 2.** We have that  $u_x = \frac{1}{2}(x' - x) = \frac{1}{2}(-\ell_{x_+} + \ell_x \ell_{x_+} \ell_x^{-1})$ , and

$$u_x \cdot \ell_x = \frac{1}{2}(u_x \ell_x + \ell_x u_x) = \frac{1}{4}(-\ell_{x_+} \ell_x + \ell_x \ell_{x_+} \ell_x^{-1} \ell_x - \ell_x \ell_{x_+} + \ell_x \ell_x \ell_{x_+} \ell_x^{-1}) = 0 .$$

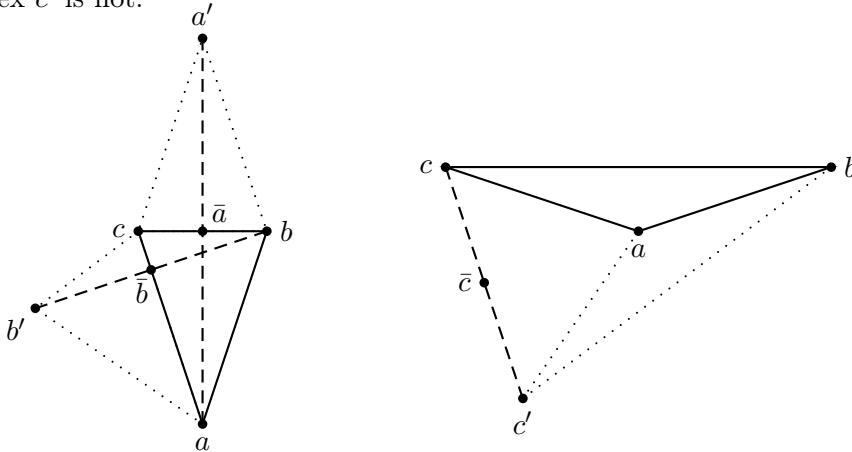
**Proof of 3.** We have that  $v_x = x_- - \frac{1}{2}(x' + x) = -\frac{1}{2}(\ell_{x_+} + \ell_x \ell_{x_+} \ell_x^{-1})$ , and

$$v_x \wedge \ell_x = \frac{1}{2}(v_x \ell_x - \ell_x v_x) = \frac{1}{4}(-\ell_{x_+} \ell_x - \ell_x \ell_{x_+} \ell_x^{-1} \ell_x + \ell_x \ell_{x_+} + \ell_x \ell_x \ell_{x_+} \ell_x^{-1}) = 0. \quad \square$$

A mirror vertex  $x'$  of the oriented triangle  $[x, x_+, x_-]$  is said to be **balanced** if  $\bar{x} \in [x_+, x_-]$ . In particular, if  $[x_+, x_-]$  is a diameter of that triangle, then  $x'$  is balanced. This implies that every triangle has at least a balanced mirror vertex. Owing to the foregoing proposition, balanced mirror vertices can be characterized through lengths.

**Proposition 3.2.** *A mirror vertex  $x'$  of the oriented triangle  $[x, x_+, x_-]$  is balanced if and only if  $|\ell_x| = |\ell_x - v_x| + |v_x|$ .*

In the two following figures, the mirror vertices  $a'$  and  $b'$  are balanced, while the mirror vertex  $c'$  is not.





## Chapter 4

# Smooth curves

In the following sections we describe some classical approximation algorithms for smooth curves in  $\mathbb{E}_n$  because of their analogies with our approximation Algorithms 6.4 and 6.7 for smooth surfaces in  $\mathbb{E}_n$ .

### 4.1 Approximation through inscribed mean vectors

Let  $I \subseteq \mathbb{R}$  be an open interval. Let  $\mathbb{E}_n$  be a  $n$ -dimensional Euclidean space, and  $\{h_1, \dots, h_n\}$  an orthonormal basis in  $\mathbb{E}_n$ .

A continuous function  $c : I \rightarrow \mathbb{E}_n$  will be simply called a **curve**.

In this chapter we define  $n$  real functions  $\gamma_j : I \rightarrow \mathbb{R}$  as the  $n$  components  $\gamma_j = c \cdot h_j$ ;

so, we have that  $\forall \tau \in I$   $c(\tau) = \sum_{j=1}^n \gamma_j(\tau) h_j$ .

A vector  $v \in \mathbb{E}_n$  is said to be **inscribed** in the curve  $c$  if there exist  $\alpha, \beta \in I$ , such that  $\alpha \neq \beta$  and  $v = c(\beta) - c(\alpha)$ ; in this case the vector  $\frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)]$  is called **inscribed mean vector** in  $c$ .

A curve  $c : I \rightarrow \mathbb{E}_n$  is said to be **smooth** if

- each  $\gamma_j$  has continuous second derivative  $\ddot{\gamma}_j$  on  $I$ , and
- there exists  $\delta > 0$  such that  $\sup_{\tau \in I} |\ddot{\gamma}_j(\tau)| \leq \delta < \infty$  ( $\delta$  does not depend on  $j = 1, \dots, n$ ).

For a smooth curve the following estimate holds for each  $\tau, \tau + \epsilon \in I$

$$|\gamma_j(\tau + \epsilon) - \gamma_j(\tau) - \dot{\gamma}_j(\tau)\epsilon| \leq \frac{\delta}{2}\epsilon^2, \quad (4.1)$$

where  $\dot{\gamma}_j$  is, of course, the first derivative of  $\gamma_j$ . Following the Landau notation, we can rewrite the relation (4.1) as follows

$$\gamma_j(\tau + \epsilon) - \gamma_j(\tau) = \dot{\gamma}_j(\tau)\epsilon + O(\epsilon^2).$$

If  $c$  is a smooth curve, we denote  $\dot{c}(\tau) = \sum_{j=1}^n \dot{\gamma}_j(\tau) h_j$ .

**Proposition 4.1.** *If  $c : I \rightarrow \mathbb{E}_n$  is a smooth curve and  $\chi \in I$ , then the vector  $\dot{c}(\chi)$  is the limit of the inscribed mean vectors  $\frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)]$  as  $(\alpha, \beta) \rightarrow (\chi, \chi)$  in  $\mathbb{R}^2$ , that is*

$$\lim_{\substack{(\alpha, \beta) \rightarrow (\chi, \chi) \\ \alpha \neq \beta}} \frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)] = \dot{c}(\chi) .$$

The proof of Proposition 4.1 is routine. Nonetheless we prove it, just because the proof we provide here is a 1-dimensional version of the proof of Theorem 6.7.

**Proof of Proposition 4.1:** let  $\alpha \neq \beta$ , then

$$\begin{aligned} \frac{\gamma_j(\beta) - \gamma_j(\alpha)}{\beta - \alpha} &= \frac{\gamma_j(\beta) - \gamma_j\left(\frac{\alpha+\beta}{2}\right) - \left[\gamma_j(\alpha) - \gamma_j\left(\frac{\alpha+\beta}{2}\right)\right]}{\beta - \alpha} = \\ &= \frac{\dot{\gamma}_j\left(\frac{\alpha+\beta}{2}\right)\left(\beta - \frac{\alpha+\beta}{2}\right) + O\left(\left(\beta - \frac{\alpha+\beta}{2}\right)^2\right)}{\beta - \alpha} - \frac{\dot{\gamma}_j\left(\frac{\alpha+\beta}{2}\right)\left(\alpha - \frac{\alpha+\beta}{2}\right) + O\left(\left(\alpha - \frac{\alpha+\beta}{2}\right)^2\right)}{\beta - \alpha} = \\ &= \dot{\gamma}_j\left(\frac{\alpha + \beta}{2}\right) + O(\beta - \alpha) . \end{aligned}$$

So,

$$\begin{aligned} \left| \frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)] - \dot{c}(\chi) \right| &= \left| \sum_{j=1}^n \left[ \dot{\gamma}_j\left(\frac{\alpha + \beta}{2}\right) - \dot{\gamma}_j(\chi) + O(\beta - \alpha) \right] h_j \right| = \\ &\leq O(\beta - \alpha) + \sum_{j=1}^n \left| \dot{\gamma}_j\left(\frac{\alpha + \beta}{2}\right) - \dot{\gamma}_j(\chi) \right| \end{aligned}$$

that goes to 0 as  $(\alpha, \beta) \rightarrow (\chi, \chi)$  because each  $\gamma_j$  is  $C^2(I)$   $\square$

## 4.2 Geometric interpretation of the direction $\dot{c}(\chi)$

If  $c$  is a smooth curve and  $\dot{c}(\chi) \neq 0$ , then  $c$  is locally injective and thus, if  $\alpha$  and  $\beta$  are sufficiently close to  $\chi \in I$ , every inscribed mean vector  $\frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)]$  is a direction, and Proposition 4.1 has the following geometric interpretation:

direction  $\dot{c}(\chi)$  is the limit of the inscribed mean directions  $\frac{1}{\beta - \alpha}[c(\beta) - c(\alpha)]$  as  $(\alpha, \beta) \rightarrow (\chi, \chi)$ .

It is well known that if  $c$  is smooth, but  $\dot{c}(\chi) = 0$ , then the foregoing interpretation may be false, even if  $c$  is locally injective. A classical example is the cusp in  $\mathbb{E}_2$ ,  $c(\tau) = \tau^2 h_1 + \tau^3 h_2$ , and  $\chi = 0$ .

### 4.3 Estimates of the length of a smooth curve

If the curve  $c : I \rightarrow \mathbb{E}_n$  is smooth and  $[\alpha, \beta] \subset I$ , then the following algorithm

$$\sum_{i=0}^k |c(\beta_i) - c(\alpha_i)| \quad (4.2)$$

can estimate the integral<sup>1</sup>  $\int_{\alpha}^{\beta} |\dot{c}(\tau)| d\tau$ , where

- $\alpha_0 = \alpha$ ,
- $\alpha_i < \alpha_{i+1} = \beta_i$  (for  $i = 0, \dots, k-1$ ), and
- $\beta_{k-1} < \beta_k = \beta$ ;

that is,  $\{[\alpha_i, \beta_i]\}_{i=0}^k = \Pi$  is a partition of  $[\alpha, \beta]$  with contiguous nonoverlapping intervals; in this sense Algorithm (4.2) can also be written

$$\sum_{[\alpha_i, \beta_i] \in \Pi} |c(\beta_i) - c(\alpha_i)| .$$

More precisely, Algorithm (4.2) converges to  $\int_{\alpha}^{\beta} |\dot{c}(\tau)| d\tau$  when the maximal length of intervals in the partition  $\Pi$ ,  $\max_{[\alpha_i, \beta_i] \in \Pi} |\beta_i - \alpha_i|$ , goes to zero, as we can see from the following elementary estimates:

$$\begin{aligned} & \left| \sum_{[\alpha_i, \beta_i] \in \Pi} |c(\beta_i) - c(\alpha_i)| - \int_{\alpha}^{\beta} |\dot{c}(\tau)| d\tau \right| = \\ &= \left| \sum_{[\alpha_i, \beta_i] \in \Pi} \left[ |c(\beta_i) - c(\alpha_i)| - \int_{\alpha_i}^{\beta_i} |\dot{c}(\tau)| d\tau \right] \right| = \\ &\leq \sum_{[\alpha_i, \beta_i] \in \Pi} \left| |c(\beta_i) - c(\alpha_i)| - \int_{\alpha_i}^{\beta_i} |\dot{c}(\tau)| d\tau \right| = \\ &= \sum_{[\alpha_i, \beta_i] \in \Pi} \left| \int_{\alpha_i}^{\beta_i} \left[ \frac{|c(\beta_i) - c(\alpha_i)|}{|\beta_i - \alpha_i|} - |\dot{c}(\tau)| \right] d\tau \right| = \\ &\leq \sum_{[\alpha_i, \beta_i] \in \Pi} \int_{\alpha_i}^{\beta_i} \left| \frac{|c(\beta_i) - c(\alpha_i)|}{|\beta_i - \alpha_i|} - |\dot{c}(\tau)| \right| d\tau = (*) . \end{aligned}$$

As  $\left| |v| - |w| \right| \leq |v - w|$  for each  $v, w \in \mathbb{E}_n$ , we have that

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<sup>1</sup>That we call **length** of the curve  $c : [\alpha, \beta] \rightarrow \mathbb{E}_n$ , when  $c$  is injective on  $[\alpha, \beta]$ .

$$\begin{aligned}
(*) &\leq \sum_{[\alpha_i, \beta_i] \in \Pi} \int_{\alpha_i}^{\beta_i} \left| \frac{1}{\beta_i - \alpha_i} [c(\beta_i) - c(\alpha_i)] - \dot{c}(\tau) \right| d\tau = \\
&= \sum_{[\alpha_i, \beta_i] \in \Pi} \int_{\alpha_i}^{\beta_i} \left| \frac{1}{\beta_i - \alpha_i} [c(\beta_i) - c(\alpha_i) - \dot{c}(\alpha_i)(\beta_i - \alpha_i)] + [\dot{c}(\alpha_i) - \dot{c}(\tau)] \right| d\tau = \\
&\leq \sum_{[\alpha_i, \beta_i] \in \Pi} \int_{\alpha_i}^{\beta_i} \left[ \left| \frac{1}{\beta_i - \alpha_i} [c(\beta_i) - c(\alpha_i) - \dot{c}(\alpha_i)(\beta_i - \alpha_i)] \right| + |\dot{c}(\alpha_i) - \dot{c}(\tau)| \right] d\tau .
\end{aligned}$$

Then, owing to estimate (4.1), we have that

$$\begin{aligned}
&\left| c(\beta_i) - c(\alpha_i) - \dot{c}(\alpha_i)(\beta_i - \alpha_i) \right| = \\
&= \left| \sum_{j=1}^n [\gamma_j(\beta_i) - \gamma_j(\alpha_i) - \dot{\gamma}_j(\alpha_i)(\beta_i - \alpha_i)] h_j \right| = \\
&\leq \sum_{j=1}^n \left| \gamma_j(\beta_i) - \gamma_j(\alpha_i) - \dot{\gamma}_j(\alpha_i)(\beta_i - \alpha_i) \right| \leq n\delta(\beta_i - \alpha_i)^2 .
\end{aligned}$$

So we can conclude that

$$\begin{aligned}
&\left| \sum_{[\alpha_i, \beta_i] \in \Pi} |c(\beta_i) - c(\alpha_i)| - \int_{\alpha}^{\beta} |\dot{c}(\tau)| d\tau \right| = \\
&\leq n\delta \sum_{[\alpha_i, \beta_i] \in \Pi} (\beta_i - \alpha_i)^2 + \sum_{[\alpha_i, \beta_i] \in \Pi} \int_{\alpha_i}^{\beta_i} |\dot{c}(\alpha_i) - \dot{c}(\tau)| d\tau .
\end{aligned}$$

That goes to zero as  $\max_{[\alpha_i, \beta_i] \in \Pi} |\beta_i - \alpha_i| \longrightarrow 0$ , since  $c$  is smooth.

Here we provided, as we did before for the proof of Proposition 4.1, many details of well-known estimates, just because of their analogies with the estimates for the area of a smooth surface<sup>2</sup>.

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<sup>2</sup>See section 6.3.



# Chapter 5

## Smooth surfaces

### 5.1 Surfaces, inscribed balanced mean bivectors

Let  $\Omega$  be an open set in  $\mathbb{E}_2$ . A continuous function  $s : \Omega \rightarrow \mathbb{E}_n$  will be simply called a **surface**.

If  $\{h_1, \dots, h_n\}$  is an ordered basis in the  $n$ -dimensional Euclidean space  $\mathbb{E}_n$  and  $s$  is a surface, we define  $n$  real functions  $\sigma_j : \Omega \rightarrow \mathbb{R}$  as the components  $\sigma_j = s \cdot h_j$ ; so, we have that  $\forall x \in \Omega \quad s(x) = \sum_{j=1}^n \sigma_j(x) h_j$ , and  $s$  is a surface if and only if each component  $\sigma_j$  is continuous. If  $\{\ell_1, \ell_2\}$  is an ordered orthonormal basis in  $\mathbb{E}_2$ , we can indicate each  $x \in \mathbb{E}_2$  as  $\chi_1 \ell_1 + \chi_2 \ell_2$ , where  $\chi_1 = x \cdot \ell_1$  and  $\chi_2 = x \cdot \ell_2$ .

**Example 5.1.** A circular right cylinder of radius  $\rho$  is a surface. Thus, it corresponds to the following function  $s : \mathbb{E}_2 \rightarrow \mathbb{E}_3$ ,

$$s(x) = s(\chi_1 \ell_1 + \chi_2 \ell_2) = \rho \cos(\chi_1) h_1 + \rho \sin(\chi_1) h_2 + \chi_2 h_3,$$

where  $\{\ell_1, \ell_2\}$  is an ordered orthonormal basis in  $\mathbb{E}_2$ ,  $\{h_1, h_2, h_3\}$  is an ordered orthonormal basis in  $\mathbb{E}_3$ .

A bivector  $V \in \mathbb{G}_{\binom{n}{2}}$  is said to be **inscribed** in a surface  $s$  if there exists a nondegenerate ordered plane triangle  $[a, b, c]$  contained in  $\Omega$ , such that

$$V = \langle s(a); s(b); s(c) \rangle = s(a) \wedge s(b) + s(b) \wedge s(c) + s(c) \wedge s(a).$$

In this case, the ordered triangle  $[s(a), s(b), s(c)]$  is said to be **inscribed** on the surface  $s$ , and the following bivector

$$\frac{1}{\langle a; b; c \rangle \cdot \mathbb{I}_2} \langle s(a); s(b); s(c) \rangle$$

is called **inscribed mean bivector** in  $s$ .

A plane triangle  $\Delta$  of vertices  $\{a, b, c\}$  is said to be **balanced** in the open set  $\Omega$  if

- $\Delta \subset \Omega$ ,
- there exists a balanced mirror vertex  $x' \in \{a', b', c'\}$  of  $\Delta$  such that the plane triangle of vertices<sup>1</sup>  $\{x', x_+, x_-\}$  is contained in  $\Omega$ .

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<sup>1</sup>See Section 3.3 for the notations.

An inscribed triangle  $[s(a), s(b), s(c)]$  is said to be **balanced** on  $s$  if the plane triangle  $[a, b, c]$  is balanced in  $\Omega$ . Note that, since  $\Omega$  is an open set, every sufficiently small<sup>2</sup> inscribed triangle  $[s(a), s(b), s(c)]$  is balanced on  $s$ .

If  $[s(a), s(b), s(c)]$  is an inscribed balanced ordered triangle on  $s$ , where  $a'$  is a balanced mirror vertex of  $[a, b, c]$  (with respect to vertex  $a$ ) such that  $[a', b, c]$  is in  $\Omega$ , then the bivector

$$[s(a') - s(a)] \wedge [s(c) - s(b)]$$

is called **inscribed balanced bivector** in  $s$ ; in this case the bivector

$$\frac{1}{2 \langle a; b; c \rangle \cdot \mathbb{I}_2} [s(a') - s(a)] \wedge [s(c) - s(b)]$$

is called **inscribed balanced mean bivector** in  $s$ .

An inscribed balanced bivector can be written in different ways

$$\begin{aligned} & [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ &= s(a) \wedge s(b) + s(b) \wedge s(a') + s(a') \wedge s(c) + s(c) \wedge s(a) = \\ &= \langle s(a); s(b); s(c) \rangle + \langle s(c); s(b); s(a') \rangle = \langle s(a); s(b); s(c) \rangle - \langle s(a'); s(b); s(c) \rangle . \end{aligned}$$

It will also be useful to write an inscribed balanced bivector coordinatewise with respect to an ordered orthonormal basis  $\{h_1, \dots, h_n\}$  in  $\mathbb{E}_n$ ; that is, if  $\forall x \in \Omega$   $s(x) = \sum_{j=1}^n \sigma_j(x) h_j$ , then

$$\begin{aligned} & [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ &= \left\{ \sum_{j=1}^n [\sigma_j(a') - \sigma_j(a)] h_j \right\} \wedge \left\{ \sum_{k=1}^n [\sigma_k(c) - \sigma_k(b)] h_k \right\} = \\ &= \sum_{1 \leq j < k \leq n} \left\{ [\sigma_j(a') - \sigma_j(a)] [\sigma_k(c) - \sigma_k(b)] - [\sigma_j(c) - \sigma_j(b)] [\sigma_k(a') - \sigma_k(a)] \right\} h_j \wedge h_k . \end{aligned}$$

Let us now define  $\binom{n}{2}$  transformations  $\mathbf{s}_{j,k} : \Omega \rightarrow \mathbb{E}_2$ ,

$$s_{j,k}(x) = \sigma_j(x) \ell_1 + \sigma_k(x) \ell_2 ;$$

then we can rewrite each component

$$\begin{aligned} & [\sigma_j(a') - \sigma_j(a)] [\sigma_k(c) - \sigma_k(b)] - [\sigma_j(c) - \sigma_j(b)] [\sigma_k(a') - \sigma_k(a)] = \\ &= \left\{ [s_{j,k}(a') - s_{j,k}(a)] \wedge [s_{j,k}(c) - s_{j,k}(b)] \right\} \cdot \mathbb{I}_2 , \end{aligned}$$

so that an inscribed balanced bivector can be written as

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<sup>2</sup>Small with respect to its diameter.

$$[s(a') - s(a)] \wedge [s(c) - s(b)] = \sum_{1 \leq j < k \leq n} \left\{ [s_{j,k}(a') - s_{j,k}(a)] \wedge [s_{j,k}(c) - s_{j,k}(b)] \right\} \cdot \mathbb{I}_2 \Big\} h_j \wedge h_k .$$

An inscribed balanced mean bivector can be written in other ways, too; in particular (as  $\langle a; b; c \rangle = \langle c; b; a' \rangle$ ) it corresponds to the following mean of inscribed mean bivectors:

$$\begin{aligned} & \frac{1}{2 \langle a; b; c \rangle \cdot \mathbb{I}_2} [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ & = \frac{1}{2} \left\{ \frac{1}{\langle a; b; c \rangle \cdot \mathbb{I}_2} \langle s(a); s(b); s(c) \rangle + \frac{1}{\langle c; b; a' \rangle \cdot \mathbb{I}_2} \langle s(c); s(b); s(a') \rangle \right\} . \end{aligned}$$

In the case of a surface in space ( $s : \Omega \rightarrow \mathbb{E}_3$ ) the foregoing mean can be seen as the mean of the vectors orthogonal to the planes secant the surfaces at points  $s(a), s(b), s(c)$  and  $s(c), s(b), s(a')$  respectively. However, the most interesting representation of an inscribed balanced bivector on  $s$  is the following:

$$\begin{aligned} & \frac{1}{2 \langle a; b; c \rangle \cdot \mathbb{I}_2} [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ & = \frac{1}{[\langle a; b; c \rangle - \langle a'; b; c \rangle] \cdot \mathbb{I}_2} \left[ \langle s(a); s(b); s(c) \rangle - \langle s(a'); s(b); s(c) \rangle \right] ; \end{aligned}$$

indeed, the previous expression plays for surfaces (in Theorem 6.7) the same role as the classical expression (for the inscribed mean vector)

$$\frac{1}{\beta - \alpha} [c(\beta) - c(\alpha)]$$

plays for curves (in Proposition 4.1).

## 5.2 Notations III

Let  $\Omega \subseteq \mathbb{E}_2$  be open. Let us give some differential notations for a sufficiently regular function  $\psi : \Omega \rightarrow \mathbb{R}$ . Let  $w \in \mathbb{E}_2$

$$\partial_{\mathbf{w}} \psi(\mathbf{x}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\psi(x + \epsilon w) - \psi(x)] \in \mathbb{R}$$

$$\nabla \psi(\mathbf{x}) = \partial_{\ell_1} \psi(x) \ell_1 + \partial_{\ell_2} \psi(x) \ell_2 \in \mathbb{E}_2 \quad \text{is the gradient vector,}$$

$$\mathbf{H}_{\psi}(\mathbf{x}) = \begin{pmatrix} \partial_{\ell_1} \partial_{\ell_1} \psi(x) & \partial_{\ell_2} \partial_{\ell_1} \psi(x) \\ \partial_{\ell_1} \partial_{\ell_2} \psi(x) & \partial_{\ell_2} \partial_{\ell_2} \psi(x) \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad \text{is the Hessian matrix,}$$

whose real eigenvalues are indicated as  $\lambda_{i, \psi(x)}$  (with  $i = 1, 2$ ), corresponding to the ordered orthonormal basis of eigenvectors  $\{\ell_{1, \psi(x)}, \ell_{2, \psi(x)}\}$  equioriented with a fixed ordered orthonormal basis  $\{\ell_1, \ell_2\}$  in  $\mathbb{E}_2 \equiv \mathbb{G}_{(2)}^2$ , that is  $\ell_{1, \psi(x)} \wedge \ell_{2, \psi(x)} = \ell_1 \wedge \ell_2 = \mathbb{I}_2$ .

### 5.3 Smooth functions, transformations and surfaces

Let  $\Omega \subseteq \mathbb{E}_2$  be open, a (real) **function**  $\psi : \Omega \rightarrow \mathbb{R}$  is said to be **smooth** if

- $\psi$  has second-order derivatives, that are continuous on  $\Omega$  (is  $C^2(\Omega)$ ),
- $\sup_{x \in \Omega} \max \{ |\lambda_{1,\psi(x)}|, |\lambda_{2,\psi(x)}| \} < +\infty$ .

For a smooth function  $\psi$  the following estimate<sup>3</sup> holds for each  $x, y \in \Omega$  such that the interval  $[x, y]$  is contained in  $\Omega$

$$|\psi(y) - \psi(x) - \nabla\psi(x) \cdot (y - x)| \leq \lambda |y - x|^2, \quad (5.1)$$

where  $\lambda = \sup_{z \in [x, y]} \max \{ |\lambda_{1,\psi(z)}|, |\lambda_{2,\psi(z)}| \}$ .

Following the Landau notation, we can rewrite relation (5.1) as follows<sup>4</sup>:

$$\forall x, y \in \Omega \quad \psi(y) - \psi(x) - \nabla\psi(x) \cdot (y - x) = O(|y - x|^2). \quad (5.2)$$

In particular, if  $z, w \in \mathbb{E}_2$  are such that the interval  $[z - w, z + w]$  is contained in  $\Omega$ , we can also write

$$\psi(z + w) - \psi(z - w) = 2\nabla\psi(z) \cdot w + O(|w|^2).$$

Let  $\Omega \subseteq \mathbb{E}_2$  be open, let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in  $\mathbb{E}_2$ , and  $\{h_1, \dots, h_n\}$  be an ordered orthonormal basis in  $\mathbb{E}_n$ ,

- a **transformation**  $f : \Omega \rightarrow \mathbb{E}_2$  is called **smooth** if each of its components  $\phi_i = f \cdot e_i$  is a smooth (real) function ( $i=1,2$ );
- a **surface**  $s : \Omega \rightarrow \mathbb{E}_n$  is called **smooth** if each of its components  $\sigma_j = s \cdot h_j$  is a smooth (real) function ( $j=1, \dots, n$ ).

**Example 5.2.** *The cylinder of example 5.1 is a smooth surface. As a matter of fact, if  $\{\ell_1, \ell_2\}$  is an ordered orthonormal basis in  $\mathbb{E}_2$  and  $x = \chi_1 \ell_1 + \chi_2 \ell_2$ , then*

$$\sigma_1(x) = \rho \cos \chi_1, \quad \sigma_2(x) = \rho \sin \chi_1, \quad \sigma_3(x) = \chi_2,$$

and  $\sup_{x \in \mathbb{E}_2} \max_{1 \leq j \leq 3} \max_{1 \leq i \leq 2} |\lambda_{i,\sigma_j(x)}| = \rho$ .

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<sup>3</sup>That estimate is analogue to estimate (4.1), and can be obtained using the Taylor formula with integral remainder.

<sup>4</sup>Here, for the sake of simplicity, we can suppose that the set  $\Omega$  is also convex.

# Chapter 6

## Main results

### 6.1 The Jacobian determinant

**Theorem 6.1.** *Let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in the Euclidean space  $\mathbb{E}_2$ ; let  $\Omega \subseteq \mathbb{E}_2$  be open; let  $f : \Omega \rightarrow \mathbb{E}_2$  be a smooth plane transformation, then  $\forall x \in \Omega$  we have that*

$$\lim_{\substack{(a,b,c) \rightarrow (x,x,x) \\ a \wedge b + b \wedge c + c \wedge a \neq 0}} \frac{\left\{ [f(d_{(a,b,c)}) - f(a)] \wedge [f(c) - f(b)] \right\} \cdot \mathbb{I}_2}{2(a \wedge b + b \wedge c + c \wedge a) \cdot \mathbb{I}_2} = (\nabla \phi_1(x) \wedge \nabla \phi_2(x)) \cdot \mathbb{I}_2 \quad (6.1)$$

where

1.  $d_{(a,b,c)} = a' = - \left[ a + 2b \frac{\ell_b \cdot \ell_a}{|\ell_a|^2} + 2c \frac{\ell_c \cdot \ell_a}{|\ell_a|^2} \right]$  is the mirror vertex of vertex  $a$ , in the oriented plane triangle  $[a, b, c]$ ,

2.  $a'$  is balanced,

and

- $\mathbb{I}_2 = \ell_1 \ell_2 = \ell_1 \wedge \ell_2$  is the pseudo-unit in the Geometric Algebra  $\mathbb{G}_2$  associated to the oriented Euclidean space  $\mathbb{E}_2$ ,
- $\wedge$  is the outer product in  $\mathbb{G}_2$ , and  $\cdot$  is the scalar product in  $\mathbb{G}_2$ ,
- $\phi_i = f \cdot \ell_i$  (with  $i = 1, 2$ ),
- the limit  $(a, b, c) \rightarrow (x, x, x)$  is taken in the product topology of  $\mathbb{E}_2 \times \mathbb{E}_2 \times \mathbb{E}_2$ .

**Remark 6.2.** *Coordinatewise, the transformation  $f$  can be seen as the real transformation  $\Phi : \tilde{\Omega} \rightarrow \mathbb{R}^2$ , where*

- $\mathbb{R}^2 \supseteq \tilde{\Omega} \ni (\chi_1, \chi_2) \iff \chi_1 \ell_1 + \chi_2 \ell_2 \in \Omega \subseteq \mathbb{E}_2$ ;
- $\Phi(\chi_1, \chi_2) = (f(\chi_1 \ell_1 + \chi_2 \ell_2) \cdot \ell_1, f(\chi_1 \ell_1 + \chi_2 \ell_2) \cdot \ell_2)$   
 $= (\phi_1(\chi_1 \ell_1 + \chi_2 \ell_2), \phi_2(\chi_1 \ell_1 + \chi_2 \ell_2));$

then, we have that

$$\begin{pmatrix} \partial_{\ell_1} \phi_1(x) & \partial_{\ell_2} \phi_1(x) \\ \partial_{\ell_1} \phi_2(x) & \partial_{\ell_2} \phi_2(x) \end{pmatrix} = \frac{\partial(\phi_1, \phi_2)}{\partial(\chi_1, \chi_2)}$$

is the Jacobian matrix of transformation  $\Phi$ , whose determinant<sup>1</sup> is

$$(\nabla \phi_1(x) \wedge \nabla \phi_2(x)) \cdot \mathbb{I}_2 = \det \frac{\partial(\phi_1, \phi_2)}{\partial(\chi_1, \chi_2)}.$$

**Remark 6.3.** It is possible to obtain an analogue result for transformations within  $k$ -dimensional Euclidean spaces. However, such result will be treated in other works, were we will apply it to construct  $k$ -dimensional Stieltjes-like measures in  $\mathbb{E}_n$ .

**Remark 6.4.** The approximating ratio in the foregoing theorem can be rewritten as a kind of incremental ratio for the function  $f : \Omega \subseteq \mathbb{E}_2 \rightarrow \mathbb{E}_2$

$$\frac{\{[f(a') - f(a)] \wedge [f(c) - f(b)]\} \cdot \mathbb{I}_2}{2(a \wedge b + b \wedge c + c \wedge a) \cdot \mathbb{I}_2} = \frac{[\langle f(a); f(b); f(c) \rangle - \langle f(a'); f(b); f(c) \rangle] \cdot \mathbb{I}_2}{[\langle a; b; c \rangle - \langle a'; b; c \rangle] \cdot \mathbb{I}_2}$$

where  $\langle x; y; z \rangle = x \wedge y + y \wedge z + z \wedge x$ . This strengthens the analogy between the derivative of a function of one real variable and the Jacobian determinant of a transformation of two real variables.

**Remark 6.5.** The coordinatewise writing of the previous approximating ratio is more elaborate and lengthy. Let us define  $\tilde{\phi}_i(\chi_1, \chi_2) = \phi_i(\chi_1 \ell_1 + \chi_2 \ell_2)$ , then  $\tilde{\phi}_i : \tilde{\Omega} \rightarrow \mathbb{R}$ . If we denote  $a = \alpha_1 \ell_1 + \alpha_2 \ell_2$ ,  $a' = \alpha'_1 \ell_1 + \alpha'_2 \ell_2$ ,  $b = \beta_1 \ell_1 + \beta_2 \ell_2$ ,  $c = \gamma_1 \ell_1 + \gamma_2 \ell_2$ , the thesis of Theorem 6.1 becomes

$$\frac{\det \begin{pmatrix} \tilde{\phi}_1(\alpha'_1, \alpha'_2) - \tilde{\phi}_1(\alpha_1, \alpha_2) & \tilde{\phi}_2(\alpha'_1, \alpha'_2) - \tilde{\phi}_2(\alpha_1, \alpha_2) \\ \tilde{\phi}_1(\gamma_1, \gamma_2) - \tilde{\phi}_1(\beta_1, \beta_2) & \tilde{\phi}_2(\gamma_1, \gamma_2) - \tilde{\phi}_2(\beta_1, \beta_2) \end{pmatrix}}{2 \det \begin{pmatrix} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \\ \gamma_1 - \beta_1 & \gamma_2 - \beta_2 \end{pmatrix}} \longrightarrow \det \frac{\partial(\tilde{\phi}_1, \tilde{\phi}_2)}{\partial(\chi_1, \chi_2)}$$

as  $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \longrightarrow (\chi_1, \chi_2, \chi_1, \chi_2, \chi_1, \chi_2)$  in  $\mathbb{R}^6$ , where

$$\alpha'_i = - \left[ \alpha_i + 2\beta_i \frac{(\alpha_1 - \gamma_1)(\gamma_1 - \beta_1) + (\alpha_2 - \gamma_2)(\gamma_2 - \beta_2)}{(\gamma_1 - \beta_1)^2 + (\gamma_2 - \beta_2)^2} + 2\gamma_i \frac{(\beta_1 - \alpha_1)(\gamma_1 - \beta_1) + (\beta_2 - \alpha_2)(\gamma_2 - \beta_2)}{(\gamma_1 - \beta_1)^2 + (\gamma_2 - \beta_2)^2} \right].$$

The comparison between the above Cartesian expressions and the Geometric Algebraic ones should suggest why we prefer the Clifford coordinate-free language.

### **Proof of Theorem 6.1.**

As  $\mathbb{E}_2$  is locally convex, every sufficiently small triangle is balanced in  $\Omega$ . Let us write the inscribed balanced bivector with respect to the ordered basis  $\{\ell_1, \ell_2\}$ ,

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<sup>1</sup>See Section 2.7.

$$\begin{aligned}
& [f(a') - f(a)] \wedge [f(c) - f(b)] = \\
& = \left\{ [\phi_1(a') - \phi_1(a)] \ell_1 + [\phi_2(a') - \phi_2(a)] \ell_2 \right\} \wedge \left\{ [\phi_1(c) - \phi_1(b)] \ell_1 + [\phi_2(c) - \phi_2(b)] \ell_2 \right\} = \\
& = \left\{ [\phi_1(a') - \phi_1(a)] [\phi_2(c) - \phi_2(b)] - [\phi_2(a') - \phi_2(a)] [\phi_1(c) - \phi_1(b)] \right\} \mathbb{I}_2
\end{aligned}$$

putting  $\bar{a} = \frac{1}{2}(a' + a)$ ,  $u_a = \frac{1}{2}(a' - a)$ ,  $v_a = c - \bar{a}$  and  $\ell_a = c - b$  we have that  $a' = \bar{a} + u_a$ ,  $a = \bar{a} - u_a$ ,  $c = \bar{a} + v_a$  and  $b = \bar{a} - (\ell_a - v_a)$ . As  $\mathbb{E}_2$  is locally convex, there exists in  $\Omega$  a convex open neighborhood of  $x$  where we can use estimate (5.2)

$$\phi_i(a') - \phi_i(a) = \phi_i(\bar{a} + u_a) - \phi_i(\bar{a} - u_a) = \nabla \phi_i(\bar{a}) \cdot (2u_a) + O(|u_a|^2)$$

$$\begin{aligned}
\phi_i(c) - \phi_i(b) &= \phi_i(\bar{a} + v_a) - \phi_i(\bar{a} - (\ell_a - v_a)) = \\
&= \phi_i(\bar{a} + v_a) - \phi_i(\bar{a}) - [\phi_i(\bar{a} - (\ell_a - v_a)) - \phi_i(\bar{a})] = \\
&= \nabla \phi_i(\bar{a}) \cdot v_a + O(|v_a|^2) + \nabla \phi_i(\bar{a}) \cdot (\ell_a - v_a) + O(|\ell_a - v_a|^2) = \\
&= \nabla \phi_i(\bar{a}) \cdot \ell_a + O(|v_a|^2) + O(|\ell_a - v_a|^2).
\end{aligned}$$

Then,

$$\begin{aligned}
& \left\{ [f(a') - f(a)] \wedge [f(c) - f(b)] \right\} \cdot \mathbb{I}_2 = \\
& = [\phi_1(a') - \phi_1(a)] [\phi_2(c) - \phi_2(b)] - [\phi_2(a') - \phi_2(a)] [\phi_1(c) - \phi_1(b)] = \\
& = [\nabla \phi_1(\bar{a}) \cdot (2u_a) + O(|u_a|^2)] [\nabla \phi_2(\bar{a}) \cdot \ell_a + O(|v_a|^2) + O(|\ell_a - v_a|^2)] + \\
& \quad - [\nabla \phi_2(\bar{a}) \cdot (2u_a) + O(|u_a|^2)] [\nabla \phi_1(\bar{a}) \cdot \ell_a + O(|v_a|^2) + O(|\ell_a - v_a|^2)] = \\
& = [\nabla \phi_1(\bar{a}) \cdot (2u_a)] [\nabla \phi_2(\bar{a}) \cdot \ell_a] - [\nabla \phi_2(\bar{a}) \cdot (2u_a)] [\nabla \phi_1(\bar{a}) \cdot \ell_a] + \\
& \quad + [\nabla \phi_1(\bar{a}) \cdot (2u_a)] [O(|v_a|^2) + O(|\ell_a - v_a|^2)] + O(|u_a|^2) [\nabla \phi_2(\bar{a}) \cdot \ell_a] + \\
& \quad + [\nabla \phi_2(\bar{a}) \cdot (2u_a)] [O(|v_a|^2) + O(|\ell_a - v_a|^2)] + O(|u_a|^2) [\nabla \phi_1(\bar{a}) \cdot \ell_a] + \\
& \quad + O(|u_a|^2) [O(|v_a|^2) + O(|\ell_a - v_a|^2)].
\end{aligned}$$

The first difference is a scalar product between bivectors in  $\mathbb{G}_2$  (i.e. pseudo-scalars)

$$\begin{aligned}
& [\nabla \phi_1(\bar{a}) \cdot (2u_a)] [\nabla \phi_2(\bar{a}) \cdot \ell_a] - [\nabla \phi_2(\bar{a}) \cdot (2u_a)] [\nabla \phi_1(\bar{a}) \cdot \ell_a] = \\
& = (\nabla \phi_1(\bar{a}) \wedge \nabla \phi_2(\bar{a})) \cdot ((2u_a) \wedge \ell_a).
\end{aligned}$$

Since  $a'$  is a mirror vertex, then  $u_a$  is a direction orthogonal to the direction  $\ell_a$ , and we have that

$$(\nabla \phi_1(\bar{a}) \wedge \nabla \phi_2(\bar{a})) \cdot ((2u_a) \wedge \ell_a) = \pm 2|u_a||\ell_a| (\nabla \phi_1(\bar{a}) \wedge \nabla \phi_2(\bar{a})) \cdot \mathbb{I}_2,$$

the sign depending on whether or not the ordered basis  $\{u_a, \ell_a\}$  is equi-oriented with the ordered orthonormal basis  $\{\ell_1, \ell_2\}$ .

Now, we simply observe that

$$2(a \wedge b + b \wedge c + c \wedge a) = (2u_a) \wedge \ell_a = \pm 2|u_a||\ell_a| \mathbb{I}_2 ,$$

and so we can write

$$\begin{aligned} & \left| \frac{\left\{ [f(a') - f(a)] \wedge [f(c) - f(b)] \right\} \cdot \mathbb{I}_2}{2(a \wedge b + b \wedge c + c \wedge a) \cdot \mathbb{I}_2} - \left( \nabla \phi_1(x) \wedge \nabla \phi_2(x) \right) \cdot \mathbb{I}_2 \right| = \\ & \leq \left| \left( \nabla \phi_1(\bar{a}) \wedge \nabla \phi_2(\bar{a}) \right) \cdot \mathbb{I}_2 - \left( \nabla \phi_1(x) \wedge \nabla \phi_2(x) \right) \cdot \mathbb{I}_2 \right| + \\ & + |\nabla \phi_1(\bar{a})| \left| \frac{O(|v_a|^2)}{|\ell_a|} + \frac{O(|\ell_a - v_a|^2)}{|\ell_a|} \right| + |\nabla \phi_2(\bar{a})| |O(|u_a|)| + \\ & + |\nabla \phi_2(\bar{a})| \left| \frac{O(|v_a|^2)}{|\ell_a|} + \frac{O(|\ell_a - v_a|^2)}{|\ell_a|} \right| + |\nabla \phi_1(\bar{a})| |O(|u_a|)| + \\ & + O(|u_a|) \left| \frac{O(|v_a|^2)}{|\ell_a|} + \frac{O(|\ell_a - v_a|^2)}{|\ell_a|} \right| \end{aligned} \quad (6.2)$$

By Cauchy-Schwarz inequality, triangular inequality and Proposition 2.1, we have that  $\forall t, w, y, z \in \mathbb{E}_2$

$$\begin{aligned} & |(t \wedge w) \cdot \mathbb{I}_2 - (y \wedge z) \cdot \mathbb{I}_2| = |t \wedge w - y \wedge z| = |t \wedge w - t \wedge z + t \wedge z - y \wedge z| = \\ & \leq |t \wedge (w - z)| + |(t - y) \wedge z| \leq |t||w - z| + |t - y||z| \end{aligned}$$

so

$$\begin{aligned} & \left| \left( \nabla \phi_1(\bar{a}) \wedge \nabla \phi_2(\bar{a}) \right) \cdot \mathbb{I}_2 - \left( \nabla \phi_1(x) \wedge \nabla \phi_2(x) \right) \cdot \mathbb{I}_2 \right| = \\ & \leq |\nabla \phi_1(\bar{a})| |\nabla \phi_2(\bar{a}) - \nabla \phi_2(x)| + |\nabla \phi_1(\bar{a}) - \nabla \phi_1(x)| |\nabla \phi_2(x)| \end{aligned}$$

Moreover, the mirror vertex  $a'$  is balanced, and then (by Proposition 3.2) we have that

$$\max \left\{ \frac{|v_a|}{|\ell_a|}, \frac{|\ell_a - v_a|}{|\ell_a|} \right\} \leq 1 . \quad (6.3)$$

As  $f$  is smooth, the theorem is proved; in fact,

$$\bar{a} \longrightarrow x \quad \text{and} \quad |u_a|, |v_a|, |\ell_a - v_a|, |\ell_a| \longrightarrow 0 ,$$

as  $(a, b, c) \longrightarrow (x, x, x)$  .  $\square$

**Remark 6.6.** As we have warned in the introduction<sup>2</sup> some hypotheses in the foregoing theorem can be weakened. For instance, we can relax hypothesis (2.) by imposing that

$$\max \left\{ \frac{|v_a|}{|\ell_a|}, \frac{|\ell_a - v_a|}{|\ell_a|} \right\}$$

be simply bounded, instead of supposing  $a'$  being balanced (that is equivalent to (6.3)). We will show in section 7.2 that even when  $d = d_{(a,b,c)}$  is not a mirror vertex of  $[a, b, c]$ , it is possible to choose suitable oriented triangles  $[d, c, b]$ , adjacent to the converging triangles  $[a, b, c]$ , such that (6.1) and (6.4) hold.

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<sup>2</sup>See Section 1.2



## 6.2 The tangent bivector

The following theorem is to smooth surfaces as Proposition 4.1 is to smooth curves.

**Theorem 6.7.** *Let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in the Euclidean space  $\mathbb{E}_2$ ; let  $\{h_1, \dots, h_n\}$  be an ordered orthonormal basis in the  $n$ -dimensional Euclidean space  $\mathbb{E}_n$ ; let  $\Omega \subseteq \mathbb{E}_2$  be open, and  $s : \Omega \rightarrow \mathbb{E}_n$  be a smooth surface, then  $\forall x \in \Omega$  we have that*

$$\lim_{\substack{(a,b,c) \rightarrow (x,x,x) \\ a \wedge b + b \wedge c + c \wedge a \neq 0}} \frac{1}{[\langle a; b; c \rangle - \langle d_{(a,b,c)}; b; c \rangle] \cdot \mathbb{I}_2} \left[ \langle s(a); s(b); s(c) \rangle - \langle s(d_{(a,b,c)}); s(b); s(c) \rangle \right] = \\ = \partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x) , \quad (6.4)$$

where

- $d_{(a,b,c)} = a' = - \left[ a + 2b \frac{\ell_b \cdot \ell_a}{|\ell_a|^2} + 2c \frac{\ell_c \cdot \ell_a}{|\ell_a|^2} \right]$  is a balanced mirror vertex of the oriented plane triangle  $[a, b, c]$ ,
- $\mathbb{I}_2 = \ell_1 \ell_2 = \ell_1 \wedge \ell_2$  is the pseudo-unit in the Geometric Algebra  $\mathbb{G}_2$  associated to the oriented Euclidean space  $\mathbb{E}_2$ ,
- $\wedge$  is the outer product in  $\mathbb{G}_k$ , and  $\cdot$  is the scalar product in  $\mathbb{G}_k$  (with  $k = 2$  or  $k = n$ ),
- $\partial_{\ell_i} \mathbf{s}(\mathbf{x}) = \sum_{j=1}^n \partial_{\ell_i} \sigma_j(x) h_j$  (with  $i = 1, 2$ ), where  $\sigma_j = s \cdot h_j$  (with  $j = 1, \dots, n$ ),
- the limit  $(a, b, c) \rightarrow (x, x, x)$  is taken in the product topology of  $\mathbb{E}_2 \times \mathbb{E}_2 \times \mathbb{E}_2$ .

### Proof of Theorem 6.7.

The proof is just a coordinatewise application of Theorem 6.1. Let us rewrite

$$\begin{aligned} & \langle s(a); s(b); s(c) \rangle - \langle s(a'); s(b); s(c) \rangle = [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ & = \left\{ \sum_{j=1}^n [\sigma_j(a') - \sigma_j(a)] h_j \right\} \wedge \left\{ \sum_{k=1}^n [\sigma_k(c) - \sigma_k(b)] h_k \right\} = \\ & = \sum_{1 \leq j < k \leq n} \left\{ [\sigma_j(a') - \sigma_j(a)] [\sigma_k(c) - \sigma_k(b)] - [\sigma_j(c) - \sigma_j(b)] [\sigma_k(a') - \sigma_k(a)] \right\} h_j \wedge h_k = \\ & = \sum_{1 \leq j < k \leq n} \left\{ \left\{ [s_{j,k}(a') - s_{j,k}(a)] \wedge [s_{j,k}(c) - s_{j,k}(b)] \right\} \cdot \mathbb{I}_2 \right\} h_j \wedge h_k \end{aligned} \quad (6.5)$$

where  $s_{j,k} = \sigma_j \ell_1 + \sigma_k \ell_2 : \Omega \rightarrow \mathbb{E}_2$  are  $\binom{n}{2}$  smooth transformations. By Theorem 6.1 we have that

$$\lim_{\substack{(a,b,c) \rightarrow (x,x,x) \\ a \wedge b + b \wedge c + c \wedge a \neq 0}} \frac{\left\{ [s_{j,k}(a') - s_{j,k}(a)] \wedge [s_{j,k}(c) - s_{j,k}(b)] \right\} \cdot \mathbb{I}_2}{2(a \wedge b + b \wedge c + c \wedge a) \cdot \mathbb{I}_2} = (\nabla \sigma_j(x) \wedge \nabla \sigma_k(x)) \cdot \mathbb{I}_2 .$$

Then, the thesis follows observing that

$$\begin{aligned}
\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x) &= \left( \sum_{j=1}^n \partial_{\ell_1} \sigma_j(x) h_j \right) \wedge \left( \sum_{k=1}^n \partial_{\ell_2} \sigma_k(x) h_k \right) = \\
&= \sum_{1 \leq j < k \leq n} [\partial_{\ell_1} \sigma_j(x) \partial_{\ell_2} \sigma_k(x) - \partial_{\ell_2} \sigma_j(x) \partial_{\ell_1} \sigma_k(x)] h_j \wedge h_k = \\
&= \sum_{1 \leq j < k \leq n} [(\nabla \sigma_j(x) \wedge \nabla \sigma_k(x)) \cdot \mathbb{I}_2] h_j \wedge h_k . \quad \square
\end{aligned} \tag{6.6}$$

### 6.3 The area

**Theorem 6.8.** *Let  $P$  be a compact polygon contained in the open set  $\Omega \subseteq \mathbb{E}_2$ ; let  $s : \Omega \rightarrow \mathbb{E}_n$  be a smooth surface; let  $\Pi$  be a partition of  $P$  into a finite family of non-overlapping<sup>3</sup> nondegenerate oriented triangles  $[a_i, b_i, c_i]$  all balanced in  $\Omega$ ;*

*let  $||\Pi|| = \max_{[a_i, b_i, c_i] \in \Pi} \{|\ell_{a_i}|, |\ell_{b_i}|, |\ell_{c_i}|\}$ ; then,*

$$\lim_{||\Pi|| \rightarrow 0} \frac{1}{4} \sum_{[a_i, b_i, c_i] \in \Pi} \left| [s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)] \right| = \int_P |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| dx , \tag{6.7}$$

where  $a'_i = - \left[ a_i + 2b_i \frac{\ell_{b_i} \cdot \ell_{a_i}}{|\ell_{a_i}|^2} + 2c_i \frac{\ell_{c_i} \cdot \ell_{a_i}}{|\ell_{a_i}|^2} \right]$  is a balanced mirror vertex for  $[a_i, b_i, c_i]$ .

In particular, if  $s$  is injective, the above integral represent the area of  $s(P) \subset \mathbb{E}_n$ .

**Remark.** *In the hypothesis of the foregoing theorem we do not require the partition  $\Pi$  to be a triangulation<sup>4</sup> of  $P$  (indeed, a vertex of triangle may be interior to the side of an adjacent triangle). However, if  $\Pi$  is a triangulation of  $P$ , then the images  $s(x)$  of its vertices  $x$  are vertices of a polyhedron inscribed on  $s$ .*

#### **Proof of Theorem 6.8.**

Without loss of generality, we can suppose that all triangles  $[a_i, b_i, c_i] \in \Pi$  are equi-oriented with  $\mathbb{I}_2 = \ell_1 \wedge \ell_2$ . Then,

$$\begin{aligned}
&\left| \sum_{[a_i, b_i, c_i] \in \Pi} \frac{1}{4} \left| [s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)] \right| - \int_P |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| dx \right| = \\
&= \left| \sum_{[a_i, b_i, c_i] \in \Pi} \left[ \frac{1}{4} \left| [s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)] \right| - \int_{[a_i, b_i, c_i]} |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| dx \right] \right| = \\
&\leq \sum_{[a_i, b_i, c_i] \in \Pi} \left| \frac{1}{4} \left| [s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)] \right| - \int_{[a_i, b_i, c_i]} |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| dx \right| = (\#)
\end{aligned}$$

<sup>3</sup>Two sets are **non-overlapping** if the interiors of those two sets have empty intersection.

<sup>4</sup>A **triangulation** of  $P$  is a partition of  $P$  into a finite number of non-overlapping triangles such that no vertex of a triangle is an internal point of a side of another.

Note that the area of each triangle  $[a_i, b_i, c_i]$  can be written as  $\frac{|u_{a_i}||\ell_{a_i}|}{2}$ , so

$$\begin{aligned} (\#) &= \sum_{[a_i, b_i, c_i] \in \Pi} \left| \int_{[a_i, b_i, c_i]} \left[ \frac{[s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)]}{2|u_{a_i}||\ell_{a_i}|} - |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| \right] dx \right| = \\ &\leq \sum_{[a_i, b_i, c_i] \in \Pi} \int_{[a_i, b_i, c_i]} \left| \frac{[s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)]}{2|u_{a_i}||\ell_{a_i}|} - |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| \right| dx . \end{aligned}$$

Since  $\mathbb{G}_{\binom{n}{2}}$  is a Euclidean space, we have that  $||V| - |W|| \leq |V - W|$  for each  $V, W \in \mathbb{G}_{\binom{n}{2}}$ , so

$$\begin{aligned} &\left| \frac{[s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)]}{2|u_{a_i}||\ell_{a_i}|} - |\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| \right| = \\ &\leq \left| \frac{1}{2|u_{a_i}||\ell_{a_i}|} \{ [s(a'_i) - s(a_i)] \wedge [s(c_i) - s(b_i)] \} - \partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x) \right| = \\ &= \left| \sum_{1 \leq j < k \leq n} \left\{ \frac{1}{2|u_{a_i}||\ell_{a_i}|} [s_{j,k}(a'_i) - s_{j,k}(a_i)] \wedge [s_{j,k}(c_i) - s_{j,k}(b_i)] - \nabla \sigma_j(x) \wedge \nabla \sigma_k(x) \right\} \cdot \mathbb{I}_2 \right\} h_j \wedge h_k \right| \\ &\leq \sum_{1 \leq j < k \leq n} \left| \frac{1}{2|u_{a_i}||\ell_{a_i}|} [s_{j,k}(a'_i) - s_{j,k}(a_i)] \wedge [s_{j,k}(c_i) - s_{j,k}(b_i)] - \nabla \sigma_j(x) \wedge \nabla \sigma_k(x) \right| , \end{aligned}$$

by equations (6.5) and (6.6). Since each  $[a_i, b_i, c_i]$  is equi-oriented with  $\mathbb{I}_2$ , and we are dealing with bivectors that are also pseudo-scalars, we can write

$$\begin{aligned} &\left| \frac{1}{2|u_{a_i}||\ell_{a_i}|} [s_{j,k}(a'_i) - s_{j,k}(a_i)] \wedge [s_{j,k}(c_i) - s_{j,k}(b_i)] - \nabla \sigma_j(x) \wedge \nabla \sigma_k(x) \right| = \\ &= \left| \frac{\{ [s_{j,k}(a'_i) - s_{j,k}(a_i)] \wedge [s_{j,k}(c_i) - s_{j,k}(b_i)] \} \cdot \mathbb{I}_2}{2(a \wedge b + b \wedge c + c \wedge a) \cdot \mathbb{I}_2} - (\nabla \sigma_j(x) \wedge \nabla \sigma_k(x)) \cdot \mathbb{I}_2 \right| = \\ &\leq \left| (\nabla \sigma_j(\bar{a}_i) \wedge \nabla \sigma_k(\bar{a}_i)) \cdot \mathbb{I}_2 - (\nabla \sigma_j(x) \wedge \nabla \sigma_k(x)) \cdot \mathbb{I}_2 \right| + \\ &\quad + |\nabla \sigma_j(\bar{a}_i)| \left| \frac{O(|v_{a_i}|^2)}{|\ell_{a_i}|} + \frac{O(|\ell_{a_i} - v_{a_i}|^2)}{|\ell_{a_i}|} \right| + |\nabla \sigma_k(\bar{a}_i)| |O(|u_{a_i}|)| + \\ &\quad + |\nabla \sigma_k(\bar{a}_i)| \left| \frac{O(|v_{a_i}|^2)}{|\ell_{a_i}|} + \frac{O(|\ell_{a_i} - v_{a_i}|^2)}{|\ell_{a_i}|} \right| + |\nabla \sigma_j(\bar{a}_i)| |O(|u_{a_i}|)| + \\ &\quad + O(|u_{a_i}|) \left| \frac{O(|v_{a_i}|^2)}{|\ell_{a_i}|} + \frac{O(|\ell_{a_i} - v_{a_i}|^2)}{|\ell_{a_i}|} \right| , \end{aligned}$$

by inequality (6.2). Then, if we sum over the  $\binom{n}{2}$  indexes  $j, k$ , and if we integrate all over  $P$ , we obtain quantities that are infinitesimal with respect to  $||\Pi||$ .  $\square$

## 6.4 The graph of a smooth function

Let  $\{\ell_1, \ell_2\}$  be an ordered orthonormal basis in the Euclidean space  $\mathbb{E}_2$ ; let  $\{h_1, h_2, h_3\}$  be an ordered orthonormal basis in the three-dimensional Euclidean space  $\mathbb{E}_3$ . We can consider  $\mathbb{E}_2 \subset \mathbb{E}_3$  by identifying  $h_1 = \ell_1$  and  $h_2 = \ell_2$ . So when we have a smooth function  $\psi : \Omega \rightarrow \mathbb{R}$ , we can consider the following smooth surface  $s : \Omega \rightarrow \mathbb{E}_3$

$$s(x) = s(\chi\ell_1 + \chi_2\ell_2) = \chi\ell_1 + \chi_2\ell_2 + \psi(\chi\ell_1 + \chi_2\ell_2)h_3 = x + \psi(x)h_3 .$$

In this case we have that

$$\begin{aligned} & [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ = & (a' - a) \wedge (c - b) - \left\{ [\psi(a') - \psi(a)](c - b) - [\psi(c) - \psi(b)](a' - a) \right\} \wedge h_3 , \end{aligned}$$

$$\begin{aligned} \partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x) &= \ell_1 \wedge \ell_2 + \partial_{\ell_2} \psi(x) \ell_1 \wedge h_3 - \partial_{\ell_1} \psi(x) \ell_2 \wedge h_3 = \\ &= \ell_1 \wedge \ell_2 - (\nabla \psi(x))^* , \end{aligned}$$

that is to say,  $\nabla \psi(x) = h_3 - (\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x))^{\#} = h_3 - (\partial_{\ell_1} s(x) \times \partial_{\ell_2} s(x))$ , and

$$|\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)| = \sqrt{1 + |\nabla \psi(x)|^2} .$$

## Chapter 7

# The local Schwarz paradox

We have seen in Chapter 4 (Proposition 4.1) that the vector  $\dot{c}(\chi)$  is the limit of inscribed mean vectors

$$\frac{1}{\beta - \alpha} [c(\beta) - c(\alpha)] ,$$

as  $\alpha$  and  $\beta$  converge to  $\chi$ . In this chapter we will verify that the Schwarz Paradox has the following local formulation: inscribed mean bivectors

$$\frac{1}{\langle a; b; c \rangle \cdot \mathbb{I}_2} \langle s(a); s(b); s(c) \rangle$$

on a smooth surface  $s : \Omega \rightarrow \mathbb{E}_n$  (such as circular right cylinder) may not converge to the bivector  $\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)$ , as  $a, b$  and  $c$  converge to the point  $x \in \Omega$ . On the contrary, we have seen in Theorem 6.7 that the corresponding inscribed balanced mean bivectors

$$\begin{aligned} & \frac{1}{2 \langle a; b; c \rangle \cdot \mathbb{I}_2} [s(a') - s(a)] \wedge [s(c) - s(b)] = \\ & = \frac{1}{[\langle a; b; c \rangle - \langle a'; b; c \rangle] \cdot \mathbb{I}_2} [\langle s(a); s(b); s(c) \rangle - \langle s(a'); s(b); s(c) \rangle] , \end{aligned}$$

always converge<sup>1</sup> to the bivector  $\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x)$ . In this chapter we will verify it on the double sequences of isosceles triangles proposed by Schwarz to prove the fallacy of Serret's definition of area.

### 7.1 The Schwarz triangles

Let us consider the circular right cylinder of Example 5.1. Such a surface is smooth, and for each  $x = \chi \ell_1 + \chi \ell_2 \in \mathbb{E}_2$

$$\partial_{\ell_1} s(x) \wedge \partial_{\ell_2} s(x) = \rho \cos(\chi_1) h_2 \wedge h_3 + \rho \sin(\chi_2) h_3 \wedge h_1 .$$

Let us consider the double sequences of oriented triangles  $[a_{m,n}, b_{m,n}, c_{m,n}]$  such that

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<sup>1</sup>As the non-degenerate triangles  $[a, b, c]$  converge to the point  $x$ .

$$a_{m,n} = 0 = x, \quad b_{m,n} = \frac{\pi}{m}\ell_1 + \frac{1}{2n}\ell_2, \quad c_{m,n} = -\frac{\pi}{m}\ell_1 + \frac{1}{2n}\ell_2.$$

Then

$$x = \lim_{m,n \rightarrow \infty} b_{m,n} = \lim_{m,n \rightarrow \infty} c_{m,n} = 0 = a_{m,n}, \quad \langle a_{m,n}; b_{m,n}; c_{m,n} \rangle = \frac{\pi}{mn} \mathbb{I}_2,$$

and

$$\langle s(a_{m,n}); s(b_{m,n}); s(c_{m,n}) \rangle = 2\rho \sin \frac{\pi}{m} \left[ \rho \left( 1 - \cos \frac{\pi}{m} \right) h_1 \wedge h_2 + \frac{1}{2n} h_2 \wedge h_3 \right]$$

so that  $\frac{1}{\langle a_{m,n}; b_{m,n}; c_{m,n} \rangle \cdot \mathbb{I}_2} \langle s(a_{m,n}); s(b_{m,n}); s(c_{m,n}) \rangle$  is asymptotically equivalent to

$$2\rho^2 n \frac{\pi^2}{m^2} h_1 \wedge h_2 + \rho h_2 \wedge h_3,$$

and then

- $\lim_{m \rightarrow \infty} \frac{1}{\langle a_{m,m}; b_{m,m}; c_{m,m} \rangle \cdot \mathbb{I}_2} \langle s(a_{m,m}); s(b_{m,m}); s(c_{m,m}) \rangle = \rho h_2 \wedge h_3 = \partial_{\ell_1} s(0) \wedge \partial_{\ell_2} s(0),$
- $\lim_{m \rightarrow \infty} \frac{1}{\langle a_{m,m^2}; b_{m,m^2}; c_{m,m^2} \rangle \cdot \mathbb{I}_2} \langle s(a_{m,m^2}); s(b_{m,m^2}); s(c_{m,m^2}) \rangle = 2\rho^2 \pi^2 h_1 \wedge h_2 + \rho h_2 \wedge h_3,$
- and the normalized direction of the mean bivector

$$\frac{1}{\langle a_{m,m^3}; b_{m,m^3}; c_{m,m^3} \rangle \cdot \mathbb{I}_2} \langle s(a_{m,m^3}); s(b_{m,m^3}); s(c_{m,m^3}) \rangle$$

tends to the planar direction  $h_1 \wedge h_2$  which is orthogonal to the tangent planar direction  $h_2 \wedge h_3$ .

On the contrary, if we consider the mirror vertex  $a'_{m,n} = \frac{1}{n}\ell_2$  (that is always balanced), we have that the balanced mean bivector

$$\frac{1}{2 \langle a; b; c \rangle \cdot \mathbb{I}_2} [s(a') - s(a)] \wedge [s(c) - s(b)] = \rho \frac{m}{\pi} \sin \frac{\pi}{m} h_2 \wedge h_3,$$

converges to the tangent bivector  $\rho h_2 \wedge h_3$ .

## 7.2 A converging mean bivector not balanced

As we have anticipated in remark 6.6

As before, we consider the Schwarz's triangles, but ordered as follows

$$a_{m,n} = -\frac{\pi}{m}\ell_1 + \frac{1}{2n}\ell_2, \quad b_{m,n} = 0 = x, \quad c_{m,n} = \frac{\pi}{m}\ell_1 + \frac{1}{2n}\ell_2.$$

As point  $d_{(a_{m,n}, b_{m,n}, c_{m,n})} = d_{m,n}$  we choose  $d_{m,n} = \frac{2\pi}{m}\ell_1$  which is not a mirror vertex of  $[a_{m,n}, b_{m,n}, c_{m,n}]$ . However, the following relation still holds

$$2 \langle a_{m,n}; b_{m,n}; c_{m,n} \rangle = \langle a_{m,n}; b_{m,n}; c_{m,n} \rangle - \langle d_{m,n}; b_{m,n}; c_{m,n} \rangle = \frac{2\pi}{mn} \mathbb{I}_2 .$$

$$\begin{aligned} & \langle s(a_{m,n}); s(b_{m,n}); s(c_{m,n}) \rangle - \langle s(d_{m,n}); s(b_{m,n}); s(c_{m,n}) \rangle = \\ &= [s(d_{m,n}) - s(a_{m,n})] \wedge [s(c_{m,n}) - s(b_{m,n})] = \\ &= \left\{ \rho \left[ \cos \left( \frac{2\pi}{m} \right) - \cos \left( \frac{\pi}{m} \right) \right] h_1 + \rho \left[ \sin \left( \frac{2\pi}{m} \right) + \sin \left( \frac{\pi}{m} \right) \right] h_2 - \frac{1}{2n} h_3 \right\} \wedge \\ & \quad \wedge \left\{ \rho \left[ \cos \left( \frac{\pi}{m} \right) - 1 \right] h_1 + \rho \sin \left( \frac{\pi}{m} \right) h_2 + \frac{1}{2n} h_3 \right\} = \\ &= \rho \frac{1}{2n} \left[ \sin \left( \frac{2\pi}{m} \right) + 2 \sin \left( \frac{\pi}{m} \right) \right] h_2 \wedge h_3 + \rho \frac{1}{2n} \left[ 1 - \cos \left( \frac{2\pi}{m} \right) \right] h_3 \wedge h_1 , \end{aligned}$$

that is asymptotically equivalent (as  $m, n \rightarrow \infty$ ) to the bivector

$$\rho \frac{2\pi}{mn} h_2 \wedge h_3 + \rho \frac{\pi^2}{m^2 n} h_3 \wedge h_1 .$$

So we can conclude that

$$\lim_{m,n \rightarrow \infty} \frac{1}{2 \langle a_{m,n}; b_{m,n}; c_{m,n} \rangle \cdot \mathbb{I}_2} [s(d_{m,n}) - s(a_{m,n})] \wedge [s(c_{m,n}) - s(b_{m,n})] = \rho h_2 \wedge h_3 .$$





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## Symbols

$X^\#$ , 12

$[a, b, c]$ , 14

$[a, b]$ , 14

$\mathbf{1}$ , 7

$\bar{a}$ , 16

$\mathbf{H}_\psi(x)$ , 25

$\dot{c}(\tau)$ , 19

$\ell_a$ , 14

$\ell_b$ , 14

$\ell_c$ , 14

$\lambda_{i,\psi(x)}$ , 25

$\langle a; b; c \rangle$ , 14

$\mathbb{O}$ , 7

$\mathbb{A}_n$ , 13

$\mathbb{E}_n$ , 3

$\mathbb{G}_n$ , 7

$\mathbb{G}^{(n)}_k$ , 8

$\mathbb{I}_n$ , 8

$\mathcal{A}_n$ , 7

$\nabla\psi(x)$ , 25

$\partial_w\psi(x)$ , 25

$\partial_{\ell_i}s(x)$ , 31

$\square$ , 11

$|\cdots|$ , 10

$s_{j,k}$ , 24

$u_a$ , 16

$v_a$ , 16

$x'$ , 16

$x \cdot y$ , 8

$x \wedge y$ , 9

$x^*$ , 12



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